Analytical study of the optical parameters for a multiple-path telescope

M. Fernandez Guasti and A. Cornejo

A reflecting telescope arrangement is proposed in which the focal distance of the objective system is greatly enhanced without increasing the overall length of the structure. The underlying idea is to use a multiply folded optical path. This setup may be viewed as a partially periodic optical system. For small departures of the mirror confocality, analytic results in the paraxial approximation are obtained.

Key words: Telescopes, optical design.

1. Introduction

In general, the angular magnification, $\mathcal{M}$, of an optical telescope requires a large objective focal length. However, this distance is limited in practice by the size of the telescopic structure.[1] In certain systems, such as the Cassegrain system, the focal length of the primary mirror is folded by a second mirror with a negative power to increase the focal distance of the objective system. In a coude focal arrangement a long focal distance is folded by the use of auxiliary plane mirrors. Here we propose a multiply folded optical path, which may be viewed as a partially periodic optical system. Systems with two reflections at the primary mirror,[2] as well as systems with two reflections at both the primary and secondary mirrors,[3] are particularly interesting cases that have been proposed before. The magnification of this arrangement is increased while the overall length of the structure is reduced. The mathematical analysis is performed within the paraxial approximation.

Section 2 reviews the matrix formalism for the single-path arrangement; Section 3 describes the double-path arrangement; Section 4 introduces the multiple-path arrangement and Section 5 studies the focal length for an arbitrary number of paths; Section 6 establishes the separation between mirrors for a given number of paths; Section 7, divided in two subsections, deals with the diameters of the optical elements; Section 8 considers the field of view of the system; Section 9 gives the conclusions.

2. Single-Path Arrangement

Matrix ray tracing is useful for the description of ray propagation in various optical systems. The rays are described by two parameters, which are their distance from the optic axis and their slope. Optical elements or free-space propagation are described by 2×2 matrices known as *ABCD* matrices. Matrix products are associative but not commutative; these properties are frequently exploited in this formalism.[4]

For a Cassegrain arrangement, let the distance from the object to the secondary mirror plane be z_0, the primary and secondary mirrors have focal lengths f_1 and f_2, respectively, the distance between the mirrors be d, and the distance from the secondary mirror to the image plane be z_i. The object to telescope path at the secondary mirror plane, using the matrix formalism, is given by

$$\begin{bmatrix} 1 & z_0 \\ 0 & 1 \end{bmatrix}, \tag{1}$$

whereas the path from the secondary mirror plane to the primary mirror, reflection on the primary mirror, the path from the primary to the secondary mirror, and reflection on the secondary mirror is represented by

$$M_t = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}. \tag{2}$$

The free-space propagation from the object to the

M. Fernandez Guasti is with the Laboratorio de Optica Cuantica, Universidad Autonoma Metropolitana-Unidad Iztapalapa, Apartado Postal 55-534, Mexico 09340, D.F., Mexico. A. Cornejo is with Instituto Nacional de Astronomia, Optica y Electrónica, Apartado Postal 51 & 216, Tonanzintla, Puebla, C.P. 72000, Mexico.

Received 6 December 1993; revised manuscript received 4 October 1994.

0003-6935/95/162908-06$06.00/0.

© 1995 Optical Society of America.

primary mirror has been separated into two parts, i.e., z_0 and d. This procedure permits a highly symmetrical mathematical description of the multiple-path arrangement, as we see in Section 4. The distance to the image plane is given by

$$\begin{bmatrix} 1 & z_i \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Performing the multiplication of the four matrices in Eq. (2), we obtain

$$M_t = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{d}{f_1} & 2d - \dfrac{d^2}{f_1} \\ \dfrac{1}{f_2} - \dfrac{1}{f_1} - \dfrac{d}{f_1 f_2} & 1 + \dfrac{2d}{f_2} - \dfrac{d}{f_1} - \dfrac{d^2}{f_1 f_2} \end{bmatrix}. \tag{4}$$

Because the overall system should be converging, element C must be negative; this implies that

$$\frac{1}{f_2} - \frac{1}{f_1} - \frac{d}{f_1 f_2} < 0 \Rightarrow d > f_1 - f_2.$$

Let us rewrite this as

$$d = f_1 - f_2 + \Delta, \tag{5}$$

where Δ is a positive quantity representing the displacement between the foci of the mirrors. Substituting Eq. (5) in telescope matrix (4) yields

$$M_t = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \dfrac{f_2 - \Delta}{f_1} & f_1 - \dfrac{(f_2 - \Delta)^2}{f_1} \\ -\dfrac{\Delta}{f_1 f_2} & \dfrac{f_1}{f_2} + \dfrac{\Delta}{f_1} - \dfrac{\Delta^2}{f_1 f_2} \end{bmatrix},$$

where we may factor the quotient of the foci; thus

$$\begin{aligned} M_t = \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \frac{f_2}{f_1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{f_2}{f_1}\begin{bmatrix} 1 - \dfrac{\Delta}{f_2} & \left(\dfrac{f_1^2}{f_2^2} + \dfrac{2\Delta}{f_2} - \dfrac{\Delta^2}{f_2^2} - 1\right) f_2 \\ -\dfrac{\Delta}{f_2^2} & \dfrac{f_1^2}{f_2^2} + \dfrac{\Delta}{f_2} - \dfrac{\Delta^2}{f_2^2} \end{bmatrix}. \end{aligned} \tag{6}$$

The object to image plane propagation is finally represented by

$$\begin{aligned} &\begin{bmatrix} 1 & z_i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & z_0 \\ 0 & 1 \end{bmatrix} \\ &\quad = \begin{bmatrix} A + Cz_i & Az_0 + B + Cz_0 z_i + Dz_i \\ C & Cz_0 + D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}. \end{aligned}$$

For a telescopic system, the image point is dependent only on the angle of the parallel incoming beams; then $A_1 = 0 = A + Cz_i \Rightarrow A = -Cz_i$. Substituting the values for A and C, we find that the distance between mirrors (having taken the foci difference), Δ, is

$$\Delta = \frac{f_2}{1 + (z_i/f_2)}, \tag{7}$$

which depends on the position of the image plane, z_i. The overall focal distance of the system is given by the inverse of the negative of the power, which is element C of the transformation matrix. Therefore

$$\frac{1}{\mathcal{F}_t} = C = -\frac{\Delta}{f_1 f_2} = -\frac{1}{f_1[1 + (z_i/f_2)]}, \tag{8}$$

and thus the system has a longer focal length than the single primary mirror by a factor of $1 + (z_i/f_2)$. This factor is increased when the image plane is further away (larger z_i) or when the focal distance of the secondary mirror is decreased.

3. Double-Path Arrangement

Let us consider an arrangement analogous to the Cassegrain arrangement, but in which we aim for a separation between mirrors such that the object beam makes two trips between the mirrors as depicted in Fig. 1. Our purpose is to decrease the angle that the ray makes with the optic axis when it reaches the image plane, because the smaller this angle the larger the angular magnification of the system. To achieve this we reflect the beam again in the primary and secondary mirrors, such that the last reflection in the secondary mirror is closer to the optic axis.

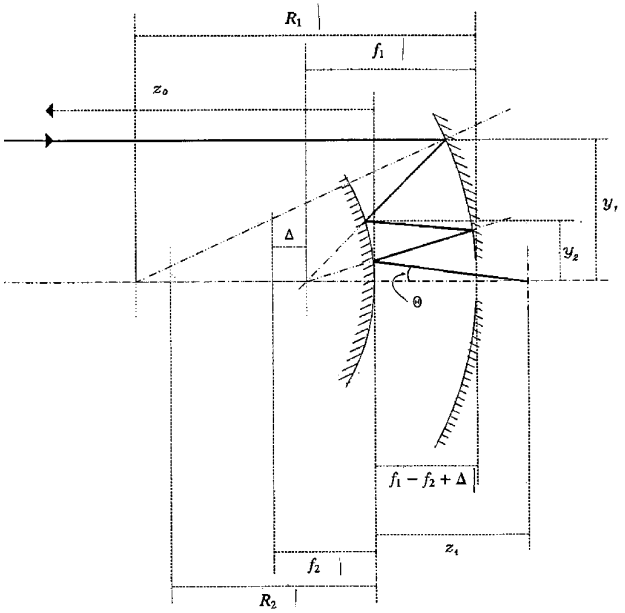

Fig. 1. Schematic arrangement for a double-path reflecting telescope: R_1, f_1, R_2, and f_2 are the radii of curvature and focal lengths of the primary and secondary mirrors, and z_0, z_i are the object and image distances from the secondary mirror plane. The foci of the mirrors are separated by distance Δ.

Another way of thinking of this problem is that, on reflection once again on the mirrors, a longer focal distance is folded in a distance defined by the separation of the two mirrors. Optical cavities in which the light encounters the same elements over and over again have been described as periodic optical systems.[4] From this point of view this system is a partially periodic system in which the number of periods is equal to 2.

Let us consider the ray-tracing description of this setup. The matrix formalism requires that we take the square of the propagation matrix, M_t:

$$M_t^2 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^2 + BC & AB + BD \\ AC + DC & CB + D^2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}.$$

This result arises from the way we display the matrices. The free-space propagation from the object to the primary mirror planes was separated as the propagation from the object to the secondary mirror plane, and from this plane to the primary mirror [relations (1) and (2)]. This procedure allows us to isolate matrix M_t as the path that is repeated on another reflection on the mirrors. In terms of the foci, the above equation yields a hardly tractable equation. Similar to the previous case, from the object to the image plane propagation we now obtain at the image plane

$$\left(\frac{f_2 - \Delta}{f_1}\right)^2\left(1 + \frac{\Delta}{f_2}\right) - \frac{\Delta}{f_2} = \left[-\frac{\Delta}{{f_1}^2{f_2}^2}({f_1}^2 + {f_2}^2 - \Delta^2)\right]z_i,$$

i.e., a cubic equation for Δ. The result obviously depends on plane z_i, where the image is focused. For three paths we would have a fifth-order polynomial, and in general for N paths we would have an $N + 2$ order polynomial. We may solve these polynomials numerically and thus obtain the separation between mirrors. However, the extraction of information from these matrices or the resulting polynomials becomes rather difficult because the complexity of the matrix elements. In the following sections we attempt to make an expansion for small departures of mirror confocality Δ so that we may obtain analytic expressions for the magnification, the separation between mirrors, and the field of view.

4. Multiple-Path Arrangement

Reflecting the rays on the mirrors several times may be repeated again to achieve even longer focal distances. Let us study the consequences of such a procedure. We call a path the trip of the ray from the secondary mirror plane to the primary mirror plane, reflection on it, propagation back to the secondary mirror plane, and reflection on it as described by matrix M_t in Eq. (6). For N paths before reaching the image plane, the arrangement is a partially periodic system with period N.

5. Overall Focal Distance

The propagation matrix is $M = M_t^N$. For an intermediate number of paths n of the total number of paths N, we have

$$M_t^n = \left(\frac{f_2}{f_1}\right)^n\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} = M_t M_t^{n-1} = \left(\frac{f_2}{f_1}\right)^n\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_{n-1} & b_{n-1} \\ c_{n-1} & d_{n-1} \end{bmatrix}. \tag{9}$$

We may write the recurrence formulas as

$$a_n = a_1 a_{n-1} + b_1 c_{n-1}, \tag{10}$$

$$c_n = c_1 a_{n-1} + d_1 c_{n-1}, \tag{11}$$

and from Eq. (6) we have

$$a_1 = 1 + f_2 c_1, \tag{12}$$

$$b_1 = (d_1 - f_2 c_1 - 1)f_2. \tag{13}$$

Therefore

$$a_n = a_{n-1} - f_2 c_{n-1} + f_2 c_n - {f_2}^2 c_1 c_{n-1},$$

where the last term on the right-hand side involves Δ quadratically or to higher order. We neglect this term because Δ is a distance parameter that becomes small as the number of paths increases. In fact, this was the reason for rewriting the distance between mirrors d in terms of the foci difference in Eq. (5). The confocal arrangement requires $d = f_1 - f_2$, which implies that $\Delta = 0$; in this case the number of paths is infinite. It is worth noting that for a single path no approximation is being made because the results are linear in Δ. We therefore have $a_n - f_2 c_n = a_{n-1} - f_2 c_{n-1}\ \forall\ n$, and from Eq. (12) we then obtain

$$a_n - f_2 c_n = 1\ \forall\ n,$$

and in particular for the last trip $a_N - f_2 c_N = 1$. Image plane z_i imposes, only for the last trip, condition $a_N = -z_i c_N$, and therefore $(z_i + f_2)c_N = -1$. However, from Eqs. (6) and (9) we find that

$$C_N = \left(\frac{f_2}{f_1}\right)^N c_N = -\left(\frac{f_2}{f_1}\right)^N\left(\frac{1}{z_i + f_2}\right).$$

The overall objective focal length, $\mathcal{F}_N$, which is proportional to angular magnification $\mathcal{M} = \mathcal{F}_N/f_c$, is equal to

$$\mathcal{F}_N = \mathcal{M} f_c = \frac{1}{C_N} = -(z_i + f_2)\left(\frac{f_1}{f_2}\right)^N = -f_1\left(1 + \frac{z_i}{f_2}\right)\left(\frac{f_1}{f_2}\right)^{N-1}, \tag{14}$$

where f_c is the eyepiece focal length. Therefore the focal length of the system increases geometrically as the number of paths increases with a factor proportional to the ratio of the primary to secondary foci.

6. Separation between Mirrors

From recurrence Eq. (11) and Eq. (12) for $n-1$, we obtain

$$c_n = c_1 + e_1 c_{n-1}, \tag{15}$$

where

$$e_1 = d_1 + f_2 c_1 = \left(\frac{f_1}{f_2}\right)^2 - \frac{\Delta^2}{{f_2}^2} \approx \left(\frac{f_1}{f_2}\right)^2,$$

and where we have neglected the quadratic Δ term on the same grounds as before. If Eq. (15) is applied for c_{n-1} then we have $c_n = c_1 + e_1(c_1 + e_1 c_{n-2})$, and by induction we find that

$$c_n = c_1 \sum_{x=0}^{s-1} {e_1}^x + {e_1}^s c_{n-s}.$$

Consider the last trip $n = N$ and let $s = N-1$; thus

$$c_N = c_1 \sum_{x=0}^{N-1} {e_1}^x \Rightarrow c_1 = c_N \left(\sum_{x=0}^{N-1} {e_1}^x\right)^{-1}.$$

Taking the value for c_N, c_1, and e_1 in terms of the foci, we obtain

$$\Delta = \frac{{f_2}^2}{(z_i + f_2)} \left[\sum_{x=0}^{N-1} \left(\frac{f_1}{f_2}\right)^{2x}\right]^{-1}.$$

Performing the sum for the geometrical series, we have

$$\Delta = \frac{{f_2}^2}{(z_i + f_2)} \frac{\left(\frac{f_1}{f_2}\right)^2 - 1}{\left(\frac{f_1}{f_2}\right)^{2N} - 1}, \tag{16}$$

where it is clear that Δ tends to zero as the number of paths tends to infinity because $f_1 > f_2$. Because the distance between mirrors is $d = f_1 - f_2 + \Delta$, as we increase the number of paths the overall length of the system is reduced. The multiple-path equation for the focal length, Eq. (14), and the distance between mirrors, Eq. (16), reduce without approximations to the usual single-path expressions (7) and (8) for $N = 1$. In Table 1 we compare the theoretical results previously derived with the numerical estimations of the beam propagation program (beams 3 and 87). Their agreement supports the validity of our derivations.

7. Diameter of the Optical Elements

The two parameters discussed here are the diameter of the secondary mirror and the diameter of the opening at the primary mirror. To this end we first calculate the distance from the axis of the nth reflection. Let the initial distance from the axis be y_1 at the plane of the primary mirror. We consider for simplicity that the ray has zero slope, $a_1 = 0$. After reflection on the primary mirror the slope of the beam is $-y_1/f_1$, and after a distance d it reaches the secondary mirror plane. The distance from axis y_2 at this plane is then

$$\frac{y_1 - y_2}{d} = \frac{y_1}{f_1}, \qquad y_2 = \left(1 - \frac{d}{f_1}\right) y_1 = \left(\frac{f_2 - \Delta}{f_1}\right) y_1.$$

After reflection on the secondary mirror, the slope is considered to be zero again. The distance from the axis at the primary mirror is then also y_2. This approximation overestimates the distance from the axis, because the slope after reflection on the secondary mirror is slightly convergent. Another reflection on the primary mirror yields a slope $-y_2/f_1$, and at the secondary mirror plane we have a distance from the axis equal to

$$y_3 = \left(\frac{f_2 - \Delta}{f_1}\right) y_2 = \left(\frac{f_2 - \Delta}{f_1}\right)^2 y_1,$$

Table 1. Comparison of Numerical and Analytical Results

	Numerical Calculation[a,b]			Analytic Prediction[b]		
Parameter	1 to 1/2	1 to 1/3	1 to 1/4	1 to /2	1 to 1/3	1 to 1/4
Single path						
$\mathcal{F}_{N=1}$	−3.0009	−4.0016	−5.0017	−3.0	−4.0	−5.0
Δ	—	—	—	0.1666	0.083333	0.05
z_i	1.0023	1.0025	1.0027	1.0	1.0	1.0
Two paths						
$\mathcal{F}_{N=2}$	−6.0072	−12.0048	−20.0133	−6.0	−12.0	−20.0
Δ	—	—	—	0.033333	0.008333	0.002941
z_i	0.9980	1.0030	1.0037	1.0	1.0	1.0
Three paths						
$\mathcal{F}_{N=3}$	−12.0048	−35.9712	80.2139	−12.0	−36.0	−80.0
Δ	—	—	—	0.007936	0.0009157	0.0001831
z_i	1.0018	1.0030	1.0050	1.0	1.0	1.0

[a] The numerical calculation requires the distance between mirrors as input, and for this purpose we use the analytical value for Δ_N from Eq. (16). The program then calculates image plane z_i as well as the overall focal length, which yield very close values to the analytic prediction.

[b] The ratios of the primary to secondary foci are $f_1 = 1.0$ and $f_2 = 0.5$ for 1 to 1/2; $f_1 = 1.0$ and $f_2 = 0.333\ldots$ for 1 to 1/3; and $f_1 = 1.0$ and $f_2 = 0.25$ for 1 to 1/4.

where again we underestimate the slope because we consider an incident beam with zero slope. By induction, for the nth path at the primary mirror plane we have

$$y_n = \left(\frac{f_2 - \Delta}{f_1}\right)^{n-1} y_1, \tag{17}$$

where the last reflection takes place in the $N - 1$ path because in the last trip the beam passes through the aperture in the primary mirror and thus misses the last reflection.

The distance from axis for the nth path reflection at the secondary mirror is

$$y_n = \left(\frac{f_2 - \Delta}{f_1}\right)^{n} y_1, \tag{18}$$

where the last reflection on the secondary mirror takes place in the last path. These functions decrease monotonically as n increases; therefore the reflections on the mirrors take place closer to the axis as the number of paths increase.

A. Diameter of the Secondary Mirror

Let D_1 be the diameter of the primary mirror. Twice the distance from the axis of a beam reflected on the rim of this mirror, at the secondary mirror plane, is

$$2y_2 = D_2 = \frac{f_2 - \Delta}{f_1} D_1,$$

and this should be the secondary mirror diameter. In terms of the number of paths, from Eq. (16) we have

$$D_2 = \left[\frac{f_2}{f_1}\right]\left[1 - \frac{f_2}{(z_i + f_2)} \frac{\left(\frac{f_1}{f_2}\right)^2 - 1}{\left(\frac{f_1}{f_2}\right)^{2N} - 1}\right] D_1. \tag{19}$$

Therefore, as the number of paths increases, the ratio of the mirrors' diameters approaches the ratio of the secondary to primary foci.

B. Diameter of the Opening in the Primary Mirror

Consider the object to telescope incoming beam with minimum distance from axis $y_{\min}$. Because $y_{\min}$ passes just by the rim of the secondary mirror, $y_{\min} = y_2$. The last reflection of this beam on the primary mirror is from Eq. (17),

$$y_N = \left(\frac{f_2 - \Delta}{f_1}\right)^{N-1} y_2,$$

and in terms of the primary diameter we have

$$2y_N = \left(\frac{f_2 - \Delta}{f_1}\right)^{N} D_1.$$

This should be the maximum diameter of the opening in the primary mirror so that this inner beam may reflect on it:

$$D_{\text{opening}} = \left[\frac{f_2}{f_1}\right]^N \left[1 - \frac{f_2}{(z_i + f_2)} \frac{\left(\frac{f_1}{f_2}\right)^2 - 1}{\left(\frac{f_1}{f_2}\right)^{2N} - 1}\right]^N D_1. \tag{20}$$

This result is an overestimate of the opening because the last reflection on the primary mirror is approximated by an upper bound.

8. Field of View

Assuming that our two-mirror telescope objective can be considered as a thin-lens objective with a stop at the lens, an incoming beam with angle θ reaches the image plane with the same angle. The image plane is located at a distance $\mathcal{F}_N$ measured from the principal plane, because the object is at infinity. Therefore

$$\theta = \frac{y_i}{\mathcal{F}_N},$$

where y_i is the distance from the axis at the image plane. Let the image plane lie close to the primary mirror plane. In this case the maximum y_i permitted is that of opening y_N, which acts as a field stop. Therefore the maximum acceptance angle, $\theta_{\max}$, is then

$$\begin{aligned}\theta_{\max} &= \frac{-1}{z_i + f_2}\left(\frac{f_2}{f_1}\right)^N \left(\frac{f_2 - \Delta}{f_1}\right)^N \\ &= -\left[\frac{f_2}{f_1}\right]^N \frac{\left[1 - \frac{f_2}{(z_i + f_2)} \frac{\left(\frac{f_1}{f_2}\right)^2 - 1}{\left(\frac{f_1}{f_2}\right)^{2N} - 1}\right]^N}{z_i + f_2} y_1.\end{aligned} \tag{21}$$

The angular aperture on the object side is twice this angle. For a large number of paths this expression reduces to

$$\lim_{N \gg 1} \theta_{\max} = \left(\frac{f_2}{f_1}\right)^{2N} \frac{-y_1}{z_i + f_2} = \left(\frac{f_2}{f_1}\right)^{2N-1} \frac{-1}{2F_1[(z_i/f_2) + 1]}, \tag{22}$$

where F_1 is the f-number of the primary mirror, $F_1 = f_1/D_1$. The field of view in this limit decreases geometrically with a factor proportional to the square of the ratio of the secondary over the primary foci.

9. Conclusions

A telescopic system with a partially periodic optical arrangement that may have some advantages over conventional systems has been described. Large angular magnifications with shortened structures is a main feature of the setup. One drawback of the proposed system is that baffling of unwanted light becomes a problem to be solved, as has been pointed out before.[3] A simple paraxial mathematical description has been made in order to stress the possibilities of the system. The approximations of the analytic

solutions show good agreement with numerical results. The shape of the surfaces of the mirrors has not been discussed and is currently subject to investigation to minimize the aberrations of the system. A 6-in. (15.24-cm) prototype is currently being tested, and there are encouraging preliminary results.

We are grateful to the reviewers for important comments and for drawing our attention to useful references.

References

1. G. P. Kuiper and B. M. Middlehurst, *Stars and Stellar Systems: Telescopes* (U. Chicago Press, Chicago, Ill., 1960), Chap. 1, p. 17.
2. D. R. Shafer, "Anastigmatic two-mirror telescopes: some new types," Appl. Opt. **16,** 1178–1180 (1977).
3. D. R. Shafer, "Large telescope designs with a spherical primary mirror," in *Instrumentation in Astronomy III,* D. L. Crawford, ed., Proc. Soc. Photo-Opt. Instrum. Eng. **172,** 19–23 (1979).
4. A. E. Siegman, *Lasers* (University Science, Mill Valley, Calif., 1986). Chap. 15, pp. 599–607.