Classroom notes

Analytic geometry of some rectilinear figures

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Some rectilinear figures, such as a square, a rectangle and a cube are described analytically through a single equation. A general equation is presented that, depending on a given parameter called the squareness parameter, yields an ellipse, a rectangle or an intermediate figure between them.

1. Introduction

Let us recall that closed conical curves arise analytically from the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where the parameters define whether we obtain a circle or an ellipse. Many other geometrical figures with smooth continuous curves may be described through a single equation. However, even very simple figures which present sharp corners, such as a square, cannot be described through a single equation, but usually have to be described analytically using four equations for straight lines limited to a certain domain.

In the present work, we develop an algorithm which permits us to describe some rectilinear figures with a single equation.

2. Equation for a square

Consider the equation

$$\left(1-\frac{x^2}{k^2}\right)^{1/2}\left(1-\frac{y^2}{k^2}\right)^{1/2}=0 \tag{2.1}$$

where x and y are real Cartesian coordinates, and k is a real constant. The solutions to the above equation are

$$x=\pm k \quad \forall y \qquad y=\pm k \quad \forall x \tag{2.2}$$

which represent two straight lines parallel to the y axis crossing the x axis at $\pm k$ and two straight lines parallel to the x axis crossing the y axis at $\pm k$ as depicted in Figure 1. If we impose the limitation that the square roots in equation (2.1) are real, this implies

$$x^2\leqslant k^2 \qquad y^2\leqslant k^2 \tag{2.3}$$

This condition restricts the infinitely long straight lines to the sides of a square measuring $2k$.

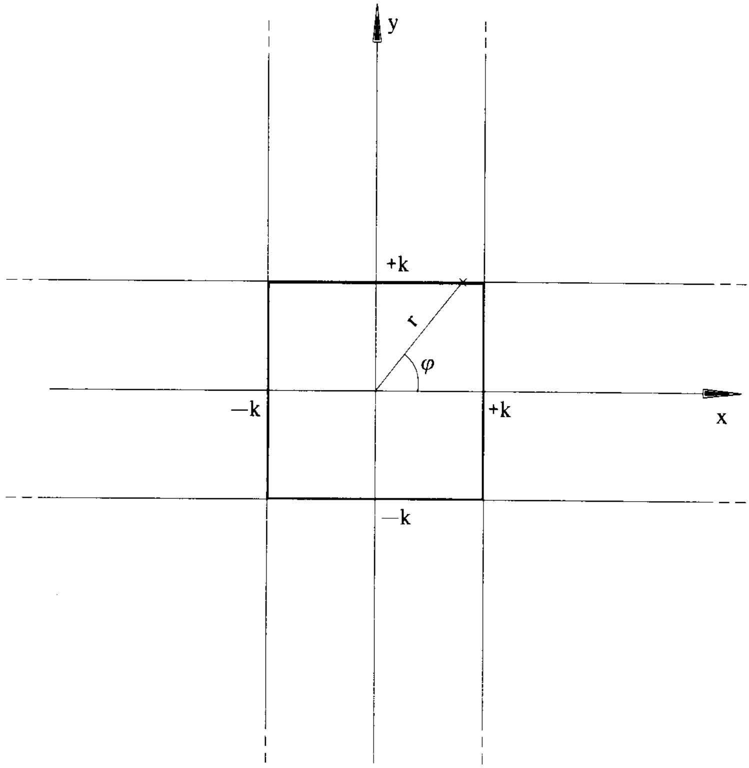

Figure 1. Solutions to equation (2.1) with (heavy line) and without (normal line) the restriction (2.3).

The polar representation of equation (2.1) may be obtained from the transformations $x=r\cos\phi$, $y=r\sin\phi$, where r is the distance from the origin to a point in the curve, and ϕ is the angle between the x axis and the line joining the origin with the point as depicted in Figure 1. The polar equation then reads

$$[\tfrac{1}{4}\sin^2(2\phi)]r^4-k^2r^2+k^4=0 \tag{2.4}$$

where the finite length condition (2.3) is imposed by

$$r^2\leqslant 2k^2 \tag{2.5}$$

The corner points of the square are obtained for the angles $\phi=\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$; the above equation is then

$$r^4-4k^2r^4+4k^4=0\Rightarrow(r^2-2k^2)^2=0\Rightarrow r=\sqrt{2}|k|$$

thus yielding the expected $\sqrt{2}$ diagonal. On the other hand, the mid-points are obtained for the angles $\phi=0,\ \pi/2,\ \pi,\ 3\pi/2$ which yield $-k^2r^2+k^4=0\Rightarrow r=k$.

Allow for the substitution $\kappa=k^2$, $\rho=r^2$, then equation (2.4) and inequality (2.5) may be rewritten as

$$[\tfrac{1}{4}\sin^2(2\phi)]\rho^2-\kappa\rho+\kappa^2=0 \qquad \rho,\kappa\geqslant 0, \qquad \rho\leqslant 2\kappa \tag{2.6}$$

The solution for ρ is then

$$\rho=\frac{\kappa}{\frac{1}{2}\sin^2(2\phi)}[1\pm(1-\sin^2(2\phi))^{1/2}] \tag{2.7}$$

Lemma 1. The negative sign of the root in the expression

$$r^2 = 2\kappa^2\left[\frac{1 \pm (1-\sin^2(2\phi))^{1/2}}{\sin^2(2\phi)}\right]$$

provides all and only all the possible solutions fulfilling the condition $r^2 \leqslant 2k^2$.

Proof. We recall that the square of the sine function is bounded between zero and one $1 \geqslant \sin^2(2\phi) \geqslant 0$, therefore

$$1 \geqslant 1-\sin^2(2\phi) \geqslant 0 \tag{2.8}$$

and thus

$$1-\sin^2(2\phi) \leqslant [1-\sin^2(2\phi)]^{1/2}$$

which rearranging terms yields

$$\frac{1-[1-\sin^2(2\phi)]^{1/2}}{\sin^2(2\phi)} \leqslant 1$$

Since the above expression is multiplied by $2k^2$ and is always less than one, it ensures condition (2.5).

On the other hand, for the positive root, we find from equation (2.8)

$$2 \geqslant 1+[1-\sin^2(2\phi)]^{1/2} \geqslant 1$$

and since the sine squared is equal to or less than one

$$\frac{1+[1-\sin^2(2\phi)]^{1/2}}{\sin^2(2\phi)} \geqslant 1$$

Thus the positive root never fulfils condition (2.5) except for the equality which is also obtained from the negative root. □

Substituting for r we obtain

$$r = \left[\frac{2k^2}{\sin^2(2\phi)}[1-(1-\sin^2(2\phi))^{1/2}]\right]^{1/2} \tag{2.9}$$

We have thus solved equation (2.4) together with the restriction imposed by (2.5). Notice that in the above expression we no longer require to impose the inequality $r^2 \leqslant 2k^2$.

We may rewrite the above solution in terms of absolute values as

$$r = \frac{\sqrt{2}|k|}{|\sin(2\phi)|}(1-|\cos(2\phi)|)^{1/2} = \begin{cases} \dfrac{k}{|\cos(\phi)|}, & -\dfrac{\pi}{4}+n\pi \quad \text{to} \quad \dfrac{\pi}{4}+n\pi \\ \dfrac{k}{|\sin(\phi)|}, & \dfrac{\pi}{4}+n\pi \quad \text{to} \quad \dfrac{3\pi}{4}+n\pi \end{cases} \tag{2.10}$$

3. Rectangle

Let us now present a more general case where the two sides of the rectilinear figure are not equal. Consider the equation

$$\left(1-\frac{x^2}{k_x^2}\right)^{1/2}\left(1-\frac{y^2}{k_y^2}\right)^{1/2} = 0 \qquad x \leqslant k_x \qquad y \leqslant k_y \tag{3.1}$$

where k_x and k_y are real constants. This equation has solutions

$$x=\pm k_x \quad \text{for} \quad y^2 \leqslant k_y^2 \qquad y=\pm k_y \quad \text{for} \quad x^2 \leqslant k_x^2 \tag{3.2}$$

which represent a rectangle whose sides are parallel to the Cartesian axes, and whose ratio is k_y/k_x.

Equation (3.1) may be recast, performing the products as

$$1-\left(\frac{x^2}{k_x^2}+\frac{y^2}{k_y^2}\right)+\frac{x^2y^2}{k_x^2k_y^2}=0 \tag{3.3}$$

The polar representations of this equation reads

$$[\tfrac{1}{4}\sin^2(2\phi)]r^4-(k_y^2\cos^2\phi+k_x^2\sin^2\phi)r^2+k_x^2k_y^2=0 \tag{3.4}$$

where

$$r^2 \leqslant k_x^2+k_y^2 \tag{3.5}$$

imposes the restricted condition stated in (3.1). Consider, for example, one of the corner points: we describe the angle in terms of the ratio of the rectangle sides. Susbtituting the identities

$$\cos^2\phi_c=\frac{1}{1+\tan^2\phi_c}=\frac{1}{1+(k_y/k_x)^2} \qquad \sin^2\phi_c=\frac{\tan^2\phi_c}{1+\tan^2\phi_c}=\frac{(k_y/k_x)^2}{1+(k_y/k_x)^2}$$

in equation (3.4) we obtain

$$r^4-2(k_y^2+k_x^2)r^2+(k_y^2+k_x^2)^2=0$$

which yields Pythagoras' theorem $r^2=k_y^2+k_x^2$.

The solution to equation (3.4) is

$$r^2=\frac{1}{2}\left(\frac{k_y}{\sin\phi}\right)^2+\frac{1}{2}\left(\frac{k_x}{\cos\phi}\right)^2-\frac{((k_y^2\cos^2\phi-k_x^2\sin^2\phi)^2)^{1/2}}{2\sin^2\phi\cos^2\phi} \tag{3.6}$$

where the negative sign of the root ensures condition (3.5), as the reader may prove in a similar way as in lemma 1. In terms of absolute values this expression may be simplified to

$$r=\frac{[(k_y\cos\phi)^2+(k_x\sin\phi)^2-|k_y^2\cos^2\phi-k_x^2\sin^2\phi|]^{1/2}}{\sqrt{2}|\sin\phi\cos\phi|}$$
$$=\begin{cases}\dfrac{k_x}{|\cos\phi|}, & -\phi_c+n\pi \quad \text{to} \quad \phi_c+n\pi \\[2ex] \dfrac{k_y}{|\sin\phi|}, & \phi_c+n\pi \quad \text{to} \quad \pi-\phi_c+n\pi\end{cases} \tag{3.7}$$

4. Closed conics and rectilinear figures

Let us return to equation (3.3), but we now introduce a squareness parameter 's^2' in the x^2y^2 product term:

$$\left(\frac{x^2}{k_x^2}+\frac{y^2}{k_y^2}\right)-\frac{s^2x^2y^2}{k_x^2k_y^2}=1 \tag{4.1}$$

When $s=0$ we obtain the equation for an ellipse with semiaxes k_x, k_y; on the other hand, when $s=1$ we obtain the equation for a rectangle with sides k_x, k_y, thus the

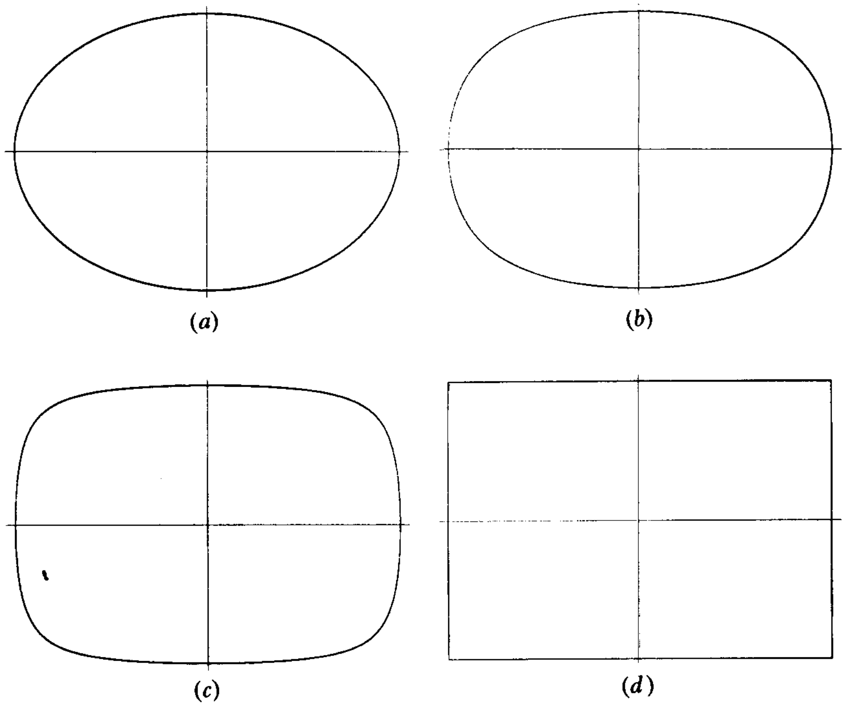

Figure 2. Solutions to equation (4.1) for different values of the squareness parameter: (*a*) $s^2=0$ yielding an ellipse with semiaxes k_x, k_y; (*b*) $s^2=0{\cdot}4$; (*c*) $s^2=0{\cdot}8$; (*d*) $s^2=1$ yielding a rectangle with half sides k_x, k_y.

ellipse is inscribed in the rectangle and their curves overlap in the x and y axes. Intermediate values of the squareness parameter yield a geometrical figure between an ellipse and a rectangle as depicted in Figures 2 (*a*)–2 (*d*). The corresponding polar equation is

$$\left[\frac{s^2}{4}\sin^2(2\phi)\right]r^4-(k_y^2\cos^2\phi+k_x^2\sin^2\phi)r^2+k_x^2k_y^2=0 \tag{4.2}$$

5. Rotation and translation

The rotation and translation of the coordinate system is given by

$$x=(x'\cos\theta+y'\sin\theta)-\alpha \qquad y=(y'\cos\theta-x'\sin\theta)-\beta \tag{5.1}$$

where the rotation angle is θ, and the new origin is (α, β).

The general equation for a rectangle is then

$$\frac{(x\cos\theta+y\sin\theta-\alpha)^2}{k_x^2}+\frac{(y\cos\theta-x\sin\theta-\beta)^2}{k_y^2}$$
$$-\frac{(x\cos\theta+y\sin\theta-\alpha)^2(y\cos\theta-x\sin\theta-\beta)^2}{k_x^2k_y^2}=1 \tag{5.2}$$

As a particular example, a square whose vertices coincide with the axes is given by the conditions $\theta=\pi/4$, $\alpha=\beta=0$, $k_x=k_y=k$ yielding

$$\frac{x^2+y^2}{k^2}-\frac{(x^2-y^2)^2}{4k^4}=1 \tag{5.3}$$

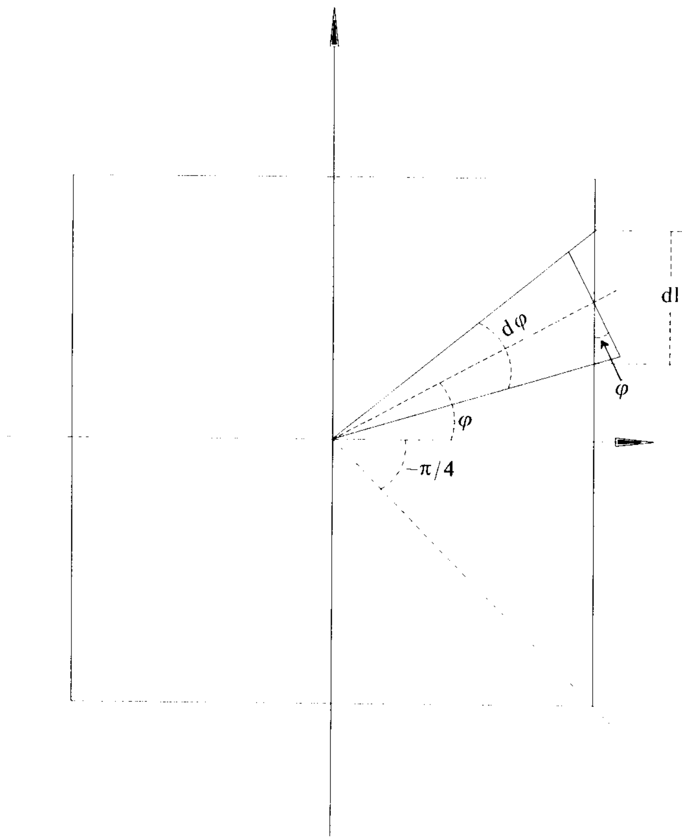

Figure 3. Differential segment and slice used to integrate in order to obtain the perimeter and the area of a square.

6. Integration

Let us consider a differential segment of the square dl. Since $r\,d\,\phi$ is a differential segment that makes an angle ϕ with the former, as shown in Figure 3, we have

$$\mathrm{d}l=\frac{r\,\mathrm{d}\phi}{\cos\phi'} \tag{6.1}$$

where ϕ' is the obliqueness angle between the y axis and the differential segment $\mathrm{d}l$; this obliqueness angle is equal to ϕ for the sides of the square parallel to the y axis, but is $\phi'=(\pi/4)-\phi$ for the sides parallel to the x axis. The perimeter l is then the integral

$$l=\int_0^{2\pi}\left(2k^2\left[\frac{1-(\cos^2(2\phi))^{1/2}}{\sin^2(2\phi)}\right]\right)^{1/2}\frac{1}{\cos\phi'}\,\mathrm{d}\phi \tag{6.2}$$

where we have used equation (2.9) to write r in terms of ϕ. We may perform the integration by separating the limits in four parts:

$$\int_{-\pi/4}^{\pi/4}+\int_{\pi/4}^{3\pi/4}+\int_{3\pi/4}^{5\pi/4}+\int_{5\pi/4}^{7\pi/4}$$

The first and second integrals, with the aid of equation (2.10) are

$$\int_{-\pi/4}^{\pi/4} \frac{k}{\cos^2\phi}\,d\phi = k\tan\phi\Big|_{-\pi/4}^{\pi/4} = 2k \qquad \int_{\pi/4}^{3\pi/4} \frac{k}{\sin^2\phi}\,d\phi = \frac{-k}{\tan\phi}\Big|_{\pi/4}^{3\pi/4} = 2k \quad (6.3)$$

Since the other two integrals are symmetrical with respect to the former two, we find that the perimeter is $l = 8k$ and integration over dk of this expression yields the area of the square $4k^2$.

Rather than integrating over strips of the perimeter, the area may also be obtained by taking pie slices. A triangular differential slice has a surface given by $ds = r(r\,d\phi)/2$, thereby eliminating the need of an obliqueness angle; the area is then

$$S = k^2\int_0^{2\pi} \frac{1-(\cos^2(2\phi))^{1/2}}{\sin^2(2\phi)}\,d\phi = k^2\int_{-\pi/4}^{\pi/4} \frac{1}{\cos^2\phi}\,d\phi + k^2\int_{\pi/4}^{3\pi/4} \frac{1}{\sin^2\phi}\,d\phi = 4k^2 \quad (6.4)$$

7. Equation for a cube

The procedure that we have introduced may be generalized to an arbitrary number of dimensions. Consider, for example, the case of a cube. The three-dimensional analogue of equation (2.1) is then

$$\left(1-\frac{x^2}{k^2}\right)^{1/2}\left(1-\frac{y^2}{k^2}\right)^{1/2}\left(1-\frac{z^2}{k^2}\right)^{1/2} = 0 \quad (7.1)$$

This equation may be rewritten in spherical coordinates as

$$[\sin^2\phi\sin^2 2\phi\sin^2 2\theta]r^6 - 4k^2[\sin^4\phi\sin^2 2\theta + \sin^2 2\phi]r^4 + 16k^4r^2 - 16k^6 = 0 \quad (7.2)$$

If $\phi = 0 \pm n\pi/2 \wedge \theta = 0 \pm n\pi/2$ then $r^2 = k^2$. If either angle equals 90 degrees, we obtain the equation for a square. It is worth mentioning that for $\theta = 0°$ we also obtain a square; however, zero degrees in azimuth yields a circle of radius 'k' because the rotation in θ is that of a line about its own axis.

The corners of a cube are placed at an angle $\theta = \pi/4$, $\sin\phi = \sqrt{2/3}$. Equation (7.2) then yields

$$r^6 - 9k^2r^4 + 27k^4r^2 - 27k^6 = 0 \quad (7.3)$$

The roots of this equation are $r^2 = 3k^2$. The diagonal to the corners are indeed $\sqrt{3}$ times the half size of a side of the cube.

8. Conclusion

Elementary analytic geometry expositions may include squares and rectangles as well as conics in their descriptions using the formalism presented here.

An application of this work is currently being undertaken in Fourier optics, introducing an aperture factor in the Fourier-Bessel transform that allows us to work any intermediate figure between a square and a circle.