# Tiered structure and symmetry of the electromagnetic equations

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**Abstract**
The structure of Maxwell's equations is reproduced for the potentials introducing a second vector potential **Ç**, in addition to the usual vector potential **A**. These in turn, can be described in terms of the so called potpotentials and the process continued ad infinitum. A unified scheme is expounded, where the relationship between contiguous and alternate tiers is presented. The Heaviside Larmor (HL) symmetry is fulfilled in any one tier. In the presence of matter, the electric-magnetic symmetry is broken by electric monopoles but preserved for the electric and magnetic dipoles. The Hertz potentials will be shown to be the second tier potentials. The obtention of vector solutions from a scalar solution, will be shown to be a direct consequence of linear superposition, the HL symmetry and the structure of the electromagnetic equations for all tiers. The Debye potentials provide such solutions for spherical symmetry. The physical reality of the quantities in any one tier are discussed, in particular regarding the question of gauge transformations and physical observables.

**KEYWORDS**
Electromagnetic fields; Symmetry; Scalar and vector potentials.

## 1. Introduction

The relevance of the electromagnetic potentials has slowly but consistently shifted from an efficient tool for calculating the electromagnetic fields to a fundamental importance of the potentials in their own right. The traditionally named 'magnetic' vector potential **A** and the scalar potential $\phi_A$ prove very useful to solve radiation problems. The main reason being that the potentials inhomogeneous wave equations involve the charge and the current as their source terms. The solution is then readily written in terms of retarded Green functions. The Liénard-Wiechert potentials provide such a description for a charged particle in motion. However, the importance of these potentials is more fundamental, because the interaction Hamiltonian between the electromagnetic fields and charges, and ultimately the energy of the system, is described in terms of the particle momentum and the field's potential **A**. This formulation is a possible mathematical description in classical mechanics and electromagnetism, namely the Lagrangian or Hamilton/Jacobi formalisms and a must in quantum mechanics, where the Schrödinger or the Dirac equations make decisive use of the Hamiltonian operator. Furthermore, phenomena such as the Aharonov-Bohm effect, suggest between other

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things, the physical reality of the potential.

The vector potentials have acquired a prominent role in the description of the electromagnetic field angular momentum content in the last decade. The helicity density flow, associated with the spin angular momentum of the field, involves the fields and the potentials explicitly. There is an ongoing discussion regarding the physical significance of these quantities and its relationship with gauge invariance [1]. The controversy arises because observables cannot be functions of gauge dependent quantities, such as the potentials, since observations are made regardless of the gauge [2]. Fulfillment of the fields rotation symmetry [3], and in particular, the Heaviside-Larmor symmetry, has required the inclusion of a second vector potential $\mathbf{C}$ [4, 5]. The electric field is then equal to the curl of the $\mathbf{C}$ potential and the magnetic field is equal to minus the time derivative of $\mathbf{C}$, corresponding to the swapping of fields with respect to their definitions in terms of $\mathbf{A}$. It has been pointed out that in vacuum, the curls of the curls of the fields also obey a set of Maxwell-like equations and so on, ad infinitum. This procedure establishes an infinite hierarchy of Maxwell-like equations [6].

The Hertz vectors or polarization potentials are field transformations that are particularly well suited to solve problems involving known external polarization densities [7, 8]. This formalism provides a simple framework to tackle problems such as a linear short dipole or a magnetic loop [9]. In recent decades, there has been renovated interest in the Hertz vectors, because they can be used to obtain vector solutions to the fields wave equation, starting with a scalar wave solution. This approach was first used for solving waveguide TE and TM field modes. Haus [10] invoked a similar procedure to generate vector field solutions with two non vanishing components from a single component vector potential. These solutions were exploited by Allen et al. in the seminal paper of orbital angular momentum [11]. A similar ansatz has been implemented with the electric Hertz vector as the starting point. In this way, vector Bessel modes with three non zero components, have been derived from a scalar wave solution [12]. The way the polarization potentials are introduced is somewhat unusual. The Hertz vectors are proposed to have a form paralleling the structures of the polarization sources in the inhomogeneous wave equations for the scalar and vector potentials [13, p. 281]. Furthermore, the physical meaning of the Hertz potentials has remained somewhat obscure. It has been recently shown that the electric Hertz vector, written as the product of a scalar potential and a constant vector, arises as consequence of the transversality of the electromagnetic fields [14].

In this communication, the tiered structure of the electromagnetic equations is derived under very general considerations. The full symmetry of the vector potentials as well as the Hertz vectors or potpotentials, will be transparent in the present scheme. This unified approach evinces the relevance of a given potential depending on the sources and the symmetry of the problem. With this formalism, a vector solution can be obtained from a scalar solution either for the fields or the potentials in any tier. Expressions involving the fields and the potentials, such as the electromagnetic helicity, can then be rigorously derived. Approximate vector solutions, for example, paraxial Laguerre or Hermite Gaussian beams, can also be dealt consistently as will be shown in a forthcoming communication. Another asset is that the vector solution does not alter significantly the state of polarization of the initial constant vector that multiplies the scalar solution. This result should prove useful for structured beams and in order to obtain localized solutions. The procedure implemented here, relies primarily on the time derivatives of the fields, whereby the fields can also be written in terms of spatial derivatives (curls) even in the presence of external electric and magnetic polarizations, as well as charges and currents. The scheme is presented in section 2, where the tiered structure is described in its simplest form to grasp the main features of the mathematical structure. A general approach is undertaken in section 3, allowing for charges, currents and polarization sources in homogeneous media without specifying a gauge. The tiered structure in the Lorenz gauge is described in section 3.3. The Hertz potentials will be shown to be the second tier potentials in section 4. We shall argue, that it is not advantageous to introduce the Hertz potentials grouped together, as has become customary. The obtention of vector solutions from scalar solutions will be discussed in section 5. It will be shown that the vector nature of any of the fields or the potentials, including the polarization potentials, can be generalized in this way. Conclusions are drawn in the last section.

## 2. Sets with Maxwell equations structure

Allow for the free fields to be written as the time derivative of the vector potentials $\mathbf{A}$ and $\mathbf{Ç}$,

$$\mathbf{E} \equiv -\partial_t \mathbf{A} \text{ and } \mathbf{H} \equiv -\partial_t \mathbf{Ç}. \tag{1}$$

In vacuum, or a homogenoeus non dispersive medium, this procedure is entirely equivalent to the usual one of commencing with the curls of the potentials [15]. It has been customary to define the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ and the electric field in terms of the $\mathbf{A}$ potential. The $\mathbf{B}$, $\mathbf{E}$ choice, is a somewhat mixed representation; a more logical approach is to consider either the electric displacement $\mathbf{D}$ and magnetic induction $\mathbf{B}$ pair [16], or the electric and magnetic fields $\mathbf{E}, \mathbf{H}$. The fields as a function of the potentials is adopted in this manuscript, in contrast with previous communications where the magnetic induction was equated to $-\partial_t \mathbf{C}$. A cedilla has been added to the potential defined in terms of the magnetic field $\mathbf{H}$, to differentiate it from the induction definition. In the absence of an external magnetic polarization, $\mathbf{Ç} = \frac{1}{\mu}\mathbf{C}$. Even if natural units were used ($\mu = \varepsilon = 1$), definitions still differ in the presence of an external polarization $\mathbf{M}_{\text{ext}}$, Eq. (9). The present $\mathbf{E}, \mathbf{H}$ choice defined in (1), conveys the full symmetry of the system. To find the dependence of the field $\mathbf{E}$ in terms of $\mathbf{Ç}$, $\mathbf{H}$ is written in terms of the potential $\mathbf{Ç}$ in the Maxwell-Ampere equation, $\nabla \times \mathbf{H} = -\nabla \times \partial_t \mathbf{Ç} = \varepsilon \partial_t \mathbf{E}$, this expression is time integrated to obtain $\mathbf{E} = -\frac{1}{\varepsilon}\nabla \times \mathbf{Ç}$. Following an analogous procedure, in the Faraday induction equation, $\mathbf{E}$ is substituted in terms of the $\mathbf{A}$ potential, $\nabla \times \mathbf{E} = -\nabla \times \partial_t \mathbf{A} = -\mu \partial_t \mathbf{H}$, and thereafter time integrated, $\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}$. In all time integrations, the constant is set equal to zero. This procedure can be repeated commencing with the potentials $\mathbf{A}$, $\mathbf{Ç}$ and writing them in terms of minus the time derivative of potpotentials $\mathbf{A}_2$, $\mathbf{Ç}_2$ and so on. The first three levels are abridged in table 1.

The recursive relationship between potentials as the process is repeated, is

$$\mathbf{A}_j = -\frac{1}{\varepsilon}\nabla \times \mathbf{Ç}_{j+1} = -\partial_t \mathbf{A}_{j+1}, \quad \mathbf{Ç}_j = \frac{1}{\mu}\nabla \times \mathbf{A}_{j+1} = -\partial_t \mathbf{Ç}_{j+1}. \tag{2}$$

The potentials $\mathbf{A}$, $\mathbf{Ç}$ are considered equal to $\mathbf{A}_1$, $\mathbf{Ç}_1$. The electric field and magnetic induction can be written in three different ways in terms of the second tier potentials, called potpotentials for short in this text. These relationships correspond to the recursive expressions between the $j$ and $j+2$ variables. They depend on the choice of the two possible expressions for the fields in terms of the potentials and the two

| Fields | Potentials | Potpotentials |
|---|---|---|
| $\nabla\cdot\mathbf{E}=0,$ | $\nabla\cdot\mathbf{A}=0,$ | $\nabla\cdot\mathbf{A}_2=0,$ |
| $\nabla\cdot\mathbf{H}=0,$ | $\nabla\cdot\mathbf{Ç}=0,$ | $\nabla\cdot\mathbf{Ç}_2=0,$ |
| $\nabla\times\mathbf{H}=\varepsilon\partial_t\mathbf{E},$ | $\nabla\times\mathbf{Ç}=\varepsilon\partial_t\mathbf{A},$ | $\nabla\times\mathbf{Ç}_2=\varepsilon\partial_t\mathbf{A}_2,$ |
| $\nabla\times\mathbf{E}=-\mu\partial_t\mathbf{H}.$ | $\nabla\times\mathbf{A}=-\mu\partial_t\mathbf{Ç}.$ | $\nabla\times\mathbf{A}_2=-\mu\partial_t\mathbf{Ç}_2.$ |

| Relationship between contiguous tier quantities | |
|---|---|
| $\mathbf{E}=\begin{cases}-\frac{1}{\varepsilon}\nabla\times\mathbf{Ç},\\ -\partial_t\mathbf{A}.\end{cases}$ | $\mathbf{A}=\begin{cases}-\frac{1}{\varepsilon}\nabla\times\mathbf{Ç}_2,\\ -\partial_t\mathbf{A}_2.\end{cases}$ |
| $\mathbf{H}=\begin{cases}\frac{1}{\mu}\nabla\times\mathbf{A},\\ -\partial_t\mathbf{Ç}.\end{cases}$ | $\mathbf{Ç}=\begin{cases}\frac{1}{\mu}\nabla\times\mathbf{A}_2,\\ -\partial_t\mathbf{Ç}_2.\end{cases}$ |
| Relationship between fields and potpotentials | |
| $\mathbf{E}=\begin{cases}-\frac{1}{\mu\varepsilon}\nabla\times\nabla\times\mathbf{A}_2,\\ \frac{1}{\varepsilon}\nabla\times\partial_t\mathbf{Ç}_2,\\ \partial_t^2\mathbf{A}_2.\end{cases}$ | $\mathbf{H}=\begin{cases}-\frac{1}{\mu\varepsilon}\nabla\times\nabla\times\mathbf{Ç}_2,\\ -\frac{1}{\mu}\nabla\times\partial_t\mathbf{A}_2,\\ \partial_t^2\mathbf{Ç}_2.\end{cases}$ |

**Table 1.** Electromagnetic equations for the free fields in three contiguous tiers, beginning with the fields. The first tier vectors are the potentials and the second tier vectors are the potpotentials.

representations of the potentials in terms of potpotentials. Two of the four possible outcomes involve a spatial and a temporal derivative, these turn out to be equal. The three different results are shown in table 1. The recursive relationships for every second tier are

$$\mathbf{A}_j = -\frac{1}{\mu\varepsilon}\nabla\times\nabla\times\mathbf{A}_{j+2} = \frac{1}{\varepsilon}\nabla\times\partial_t\mathbf{Ç}_{j+2} = \partial_t^2\mathbf{A}_{j+2}, \tag{3}$$

$$\mathbf{Ç}_j = -\frac{1}{\mu\varepsilon}\nabla\times\nabla\times\mathbf{Ç}_{j+2} = -\frac{1}{\mu}\nabla\times\partial_t\mathbf{A}_{j+2} = \partial_t^2\mathbf{Ç}_{j+2}. \tag{4}$$

If both fields are written in terms of the potentials, upon time integration of the equations, a set of first order differential equations equivalent to Maxwell's electromagnetic equations are obtained. The set of electromagnetic equations for the j$^{th}$ potentials are,

$$\nabla\cdot\mathbf{A}_j = 0, \tag{5a}$$

$$\nabla\cdot\mathbf{Ç}_j = 0, \tag{5b}$$

$$\nabla\times\mathbf{Ç}_j = \varepsilon\partial_t\mathbf{A}_j, \tag{5c}$$

$$\nabla\times\mathbf{A}_j = -\mu\partial_t\mathbf{Ç}_j. \tag{5d}$$

Due to the plus sign in (5c) and minus sign in (5d), the $\mathbf{A}_j$'s emulate the role of the electric field in Maxwell's equations whereas the $\mathbf{Ç}_j$'s emulate the role of the magnetic field. Let the $\mathbb{EM}$ set in vacuum, be the set of vectors $\mathbf{A}_j, \mathbf{Ç}_j$ that satisfy equations (5a)-(5d). From the curl of (5c), (5d), it can be seen that all the vector functions satisfy homogeneous wave equations,

$$\nabla^2\mathbf{A}_j - \mu\varepsilon\partial_t^2\mathbf{A}_j = 0, \quad \nabla^2\mathbf{Ç}_j - \mu\varepsilon\partial_t^2\mathbf{Ç}_j = 0. \tag{6}$$

The electromagnetic equations for the j$^{th}$ potentials are invariant under the continuous rotation symmetry also referred to as dual symmetry in the tensor formalism [3],

$$\mathbf{A}_j \to \mathbf{A}_j\cos\theta + \sqrt{\frac{\mu}{\varepsilon}}\mathbf{Ç}_j\sin\theta,\ \mathbf{Ç}_j \to \mathbf{Ç}_j\cos\theta - \sqrt{\frac{\varepsilon}{\mu}}\mathbf{A}_j\sin\theta. \tag{7}$$

In particular, each tier satisfies the $\theta = \frac{\pi}{2}$, Heaviside-Larmor rotation symmetry. The impedance $Z = \sqrt{\frac{\mu}{\varepsilon}} = \left[C^{-1}Vs\right] = \left[C^{-2}Js\right]$, establishes the ratio of the electric over the magnetic field magnitudes $Z = \frac{|\mathbf{E}|}{|\mathbf{H}|}$. For any tier, this ratio is $Z = \frac{|\mathbf{A}_j|}{|\mathbf{Ç}_j|}$.

The sets of four first order differential equations (5a)-(5d), has its origin in the field equations. However, it could be extended in the other direction if the fields are in turn vector potentials of the hyperfields $\mathbf{A}_{-1}$ and $\mathbf{Ç}_{-1}$, such that $\mathbf{A}_{-1} = -\frac{1}{\varepsilon}\nabla\times\mathbf{H} = -\partial_t\mathbf{E}$ and $\mathbf{Ç}_{-1} = \frac{1}{\mu}\nabla\times\mathbf{E} = -\partial_t\mathbf{H}$. The relationships between two contiguous hyperfields are then $\mathbf{A}_{j-1} = -\frac{1}{\varepsilon}\nabla\times\mathbf{Ç}_j = -\partial_t\mathbf{A}_j$ and $\mathbf{Ç}_{j-1} = \frac{1}{\mu}\nabla\times\mathbf{A}_j = -\partial_t\mathbf{Ç}_j$. This result

| tier | | numbered vector functions | usual labeling |
|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | |
| -1 | hyperpotentials | $\mathbf{A}_{-1}, \mathbf{Ç}_{-1}$ | |
| 0 | fields | $\mathbf{A}_0, \mathbf{Ç}_0$ | $\mathbf{E}, \mathbf{H}$ (physical reality) |
| 1 | potentials | $\mathbf{A}_1, \mathbf{Ç}_1$ | $\mathbf{A}, \mathbf{Ç}$ |
| 2 | potpotentials (Hertz vectors) | $\mathbf{A}_2, \mathbf{Ç}_2$ | $-\mu\varepsilon\mathbf{\Pi}_e, -\varepsilon\mathbf{\Pi}_m$ |
| $\vdots$ | $\vdots$ | $\vdots$ | |

**Table 2.** Tiered structure of electromagnetic equations. Each vector function satisfies a homogeneous wave equation and each pair of vector functions satisfies Maxwell like equations.

can be generalized in (2) and (5a)-(5d), letting $j$ to acquire negative values and setting $\mathbf{E} = \mathbf{A}_0$ and $\mathbf{H} = \mathbf{Ç}_0$. These tiered sets structure is summarized in table 2. The identification of the polarization potentials follow the definitions used by various authors when introduced separately [7, 17] for free fields [18, 9, 14]. A rigorous correspondence is established in section 4. The $\mathbf{A}_2$ potpotential is again identified with the electric polarization potential and $\mathbf{Ç}_2$ with the magnetic polarization potential.

Let us remark that setting the time integration constants to zero is an important constraint [19]. The divergence equations (5a), (5b) in the Maxwell's type equations can be obtained by evaluating the divergence of the curl equations (5c), (5d), thereafter performing a time integration and setting the integration constants to zero. For this reason, the div equations have sometimes been considered redundant. However, the system is not overdetermined and the div equations are necessary, among other things, to avoid spurious modes and inaccurate solutions in computational electromagnetics [20].

From the infinite set of tiers, if the fields are monochromatic,

$$\mathbf{E} = \mathbf{A}_0 = \partial_t^2 \mathbf{A}_2 = -\omega^2 \mathbf{A}_2 = -\frac{1}{\mu\varepsilon} \nabla \times \nabla \times \mathbf{A}_2, \tag{8a}$$

$$\mathbf{H} = \mathbf{Ç}_0 = \partial_t^2 \mathbf{Ç}_2 = -\omega^2 \mathbf{Ç}_2 = -\frac{1}{\mu\varepsilon} \nabla \times \nabla \times \mathbf{Ç}_2. \tag{8b}$$

These relationships of course hold for any $\mathbf{A}_{j+2}, \mathbf{A}_j$ or $\mathbf{Ç}_{j+2}, \mathbf{Ç}_j$ pair. So, for harmonic fields, the second tier set reproduces the fields of the zeroth order set, the only difference being a $-\omega^2$ scaling factor. Therefore, for monochromatic free fields, three contiguous tiers are sufficient to establish the tiered structure of the system. Since $\mathbf{A}_j = -\partial_t \mathbf{A}_{j+1} = i\omega \mathbf{A}_{j+1}$, $\mathbf{Ç}_j = -\partial_t \mathbf{Ç}_{j+1} = i\omega \mathbf{Ç}_{j+1}$, all the $\mathbf{A}_j$ vectors are linearly dependent in the complex domain and so are all the $\mathbf{Ç}_j$ vectors. However, this is not the case in the real vectors set, where linear dependence of the $\mathbf{A}_j$'s (or $\mathbf{Ç}_j$'s) is achieved for the second time derivatives, that is, every second tier. The time derivatives of the $\mathbf{A}_j$'s and $\mathbf{Ç}_j$'svectors are related to each other by their curls. These results are relevant in order to obtain independent vector solutions to the wave equation as we shall see in section 5.

## 3. Tiered structure in matter

The general problem becomes more involved for three reasons: i) Material media are considered, to keep things tractable, the media are considered isotropic and homogeneous. ii) Sources are included in the form of electric charges and currents as well as external electric and magnetic polarizations. iii) Scalar as well as vector potentials are included, so that the gauge becomes an issue. In the early stages, we shall avoid specifying a gauge.

The electric displacement and magnetic induction for linear, isotropic media with external electric $\mathbf{P}_{\text{ext}}$ and magnetic $\mathbf{M}_{\text{ext}}$ polarization densities are [13, 21],

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_{\text{ext}}, \qquad \mathbf{B} = \mu \mathbf{H} + \mu_0 \mathbf{M}_{\text{ext}}. \tag{9}$$

The electromagnetic equations for these materials in the presence of charge $\rho$ and current $\mathbf{J}$ densities are

$$\nabla \cdot (\varepsilon \mathbf{E} + \mathbf{P}_{\text{ext}}) = \rho, \tag{10a}$$

$$\nabla \cdot (\mu \mathbf{H} + \mu_0 \mathbf{M}_{\text{ext}}) = 0, \tag{10b}$$

$$\nabla \times \mathbf{H} = \partial_t (\varepsilon \mathbf{E} + \mathbf{P}_{\text{ext}}) + \mathbf{J}, \tag{10c}$$

$$\nabla \times \mathbf{E} = -\partial_t (\mu \mathbf{H} + \mu_0 \mathbf{M}_{\text{ext}}). \tag{10d}$$

Consider for simplicity, space and time independent permittivity and permeability. The electric field wave equation in the presence of sources is

$$\nabla^2 \mathbf{E} - \mu\varepsilon \partial_t^2 \mathbf{E} = \frac{1}{\varepsilon} \nabla \rho - \frac{1}{\varepsilon} \nabla (\nabla \cdot \mathbf{P}_{\text{ext}}) + \mu \partial_t^2 \mathbf{P}_{\text{ext}} + \mu_0 \nabla \times \partial_t \mathbf{M}_{\text{ext}} + \mu \partial_t \mathbf{J}, \tag{11a}$$

and the corresponding magnetic field wave equation is

$$\nabla^2 \mathbf{H} - \mu\varepsilon \partial_t^2 \mathbf{H} = -\frac{\mu_0}{\mu} \nabla (\nabla \cdot \mathbf{M}_{\text{ext}}) + \mu_0 \varepsilon \partial_t^2 \mathbf{M}_{\text{ext}} - \nabla \times \partial_t \mathbf{P}_{\text{ext}} - \nabla \times \mathbf{J}. \tag{11b}$$

### *3.1. Potentials*

The fields in terms of the $\mathbf{A}$, $\mathbf{Ç}$ vectors and $\phi_A$, $\phi_C$ scalar potentials are [22],

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi_A, \quad \mathbf{H} = -\partial_t \mathbf{Ç} - \nabla \phi_C. \tag{12a}$$

Substitution of $\mathbf{H}$ from (12a) in the Ampere equation (10c) gives $\mathbf{E}$ in terms of $\mathbf{Ç}$. An analogous procedure is used to obtain $\mathbf{H}$ in terms of $\mathbf{A}$. The fields in terms of curls of the vector potentials are then

$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla \times \mathbf{Ç} - \frac{1}{\varepsilon}\mathbf{P}_{\text{ext}} - \frac{1}{\varepsilon}\int \mathbf{J} dt, \tag{12b}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} - \frac{\mu_0}{\mu}\mathbf{M}_{\text{ext}}. \tag{12c}$$

The potentials equations are obtained by replacement of both fields in terms of the potentials in the electromagnetic equations (10a)-(10d), followed by temporal integration and setting the integration constants equal to zero. The set of first order differential equations for the vector potentials are

$$\nabla \cdot \mathbf{A} = -\int \nabla^2 \phi_A dt + \frac{1}{\varepsilon}\int \nabla \cdot \mathbf{P}_{\text{ext}} dt - \frac{1}{\varepsilon}\int \rho dt. \tag{13a}$$

$$\nabla \cdot \mathbf{Ç} = -\int \nabla^2 \phi_C dt + \frac{\mu_0}{\mu}\int \nabla \cdot \mathbf{M}_{\text{ext}} dt. \tag{13b}$$

$$\nabla \times \mathbf{Ç} = \varepsilon \partial_t \mathbf{A} + \varepsilon \nabla \phi_A - \mathbf{P}_{\text{ext}} - \int \mathbf{J} dt. \tag{13c}$$

$$\nabla \times \mathbf{A} = -\mu \partial_t \mathbf{Ç} - \mu \nabla \phi_C + \mu_0 \mathbf{M}_{\text{ext}}. \tag{13d}$$

The source terms in (10a) and (10c), in addition to the signs in the curl equations mentioned before, suggest that $\mathbf{A}$ plays the role of $\mathbf{E}$ and $\mathbf{Ç}$ plays the role of $\mathbf{H}$ in the above set of equations. In contrast with what is customary, this state of affairs suggest that the vector and scalar potentials $\mathbf{A}$ and $\phi_A$ should be labeled the electric potentials whereas the vector and scalar potentials $\mathbf{Ç}$ and $\phi_C$ should be the magnetic potentials.

The inhomogeneous wave equation for $\mathbf{A}$ obtained from this set of first order equations is (An alternative shortcut is to use the Ampere-Maxwell equation (10c), replacing $\mathbf{H}$ with curl $\mathbf{A}$ from (12c) and $\mathbf{E}$ with the time derivative of $\mathbf{A}$ from (12a))

$$\nabla^2 \mathbf{A} - \mu\varepsilon \partial_t^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) + \mu\varepsilon \partial_t \nabla \phi_A - \mu \partial_t \mathbf{P}_{\text{ext}} - \mu_0 \nabla \times \mathbf{M}_{\text{ext}} - \mu \mathbf{J}. \tag{14a}$$

The $\nabla (\nabla \cdot \mathbf{A})$ term can of course be replaced using (13a), so that the source terms in (14a) do not involve the potential $\mathbf{A}$ explicitly. The inhomogeneous wave equation for

$\mathbf{Ç}$ is

$$\nabla^2 \mathbf{Ç} - \mu\varepsilon\partial_t^2 \mathbf{Ç} = \nabla\left(\nabla \cdot \mathbf{Ç}\right) + \mu\varepsilon\partial_t \nabla\phi_C - \mu_0\varepsilon\partial_t \mathbf{M}_{\text{ext}} + \nabla \times \mathbf{P}_{\text{ext}} + \int \nabla \times \mathbf{J} dt. \quad (14b)$$

The second order equations for the scalar potentials are,

$$\nabla^2 \phi_A + \partial_t \nabla \cdot \mathbf{A} = \frac{1}{\varepsilon}\nabla \cdot \mathbf{P}_{\text{ext}} - \frac{1}{\varepsilon}\rho, \quad (14c)$$

$$\nabla^2 \phi_C + \partial_t \nabla \cdot \mathbf{Ç} = \frac{\mu_0}{\mu}\nabla \cdot \mathbf{M}_{\text{ext}}. \quad (14d)$$

### *3.2. Potpotentials*

To follow as closely as possible the substitutions in the previous tier, the potentials $\mathbf{A}$ and $\mathbf{Ç}$ are written in terms of the time derivative of the potpotentials as

$$\mathbf{A} = -\partial_t \mathbf{A}_2 - \nabla\phi_{A2}, \quad \mathbf{Ç} = -\partial_t \mathbf{Ç}_2 - \nabla\phi_{C2}. \quad (15a)$$

To obtain $\mathbf{A}$ in terms of $\mathbf{Ç}_2$, we substitute $\mathbf{Ç}$ from (15a) in the curl of the potential EM equation (13c) and time integrate. Similarly, we find $\mathbf{Ç}$ in terms of $\mathbf{A}_2$, by substitution of $\mathbf{A}$ from (15a) in the curl of the potential EM equation (13d). The relationships between potentials and curl of potpotentials are

$$\mathbf{A} = -\frac{1}{\varepsilon}\nabla \times \mathbf{Ç}_2 - \int \nabla\phi_A dt + \frac{1}{\varepsilon}\int \left(\mathbf{P}_{\text{ext}} dt + \int \mathbf{J} dt\right) dt \quad (15b)$$

$$\mathbf{Ç} = \frac{1}{\mu}\nabla \times \mathbf{A}_2 - \int \nabla\phi_C dt + \frac{\mu_0}{\mu}\int \mathbf{M}_{\text{ext}} dt \quad (15c)$$

The fields can be written in three different ways in terms of the potpotentials depending on whether the time derivatives (12a), (15a) or the curl expressions (12b)-(12c), (15b)-(15c) are invoked. The electric field in terms of the potpotentials is

$$\mathbf{E} = -\frac{1}{\mu\varepsilon}\nabla \times \nabla \times \mathbf{A}_2 - \frac{\mu_0}{\mu\varepsilon}\int \nabla \times \mathbf{M}_{\text{ext}} dt - \frac{1}{\varepsilon}\mathbf{P}_{\text{ext}} - \frac{1}{\varepsilon}\int \mathbf{J} dt, \quad (16a)$$

$$\mathbf{E} = \frac{1}{\varepsilon}\nabla \times \partial_t \mathbf{Ç}_2 - \frac{1}{\varepsilon}\mathbf{P}_{\text{ext}} - \frac{1}{\varepsilon}\int \mathbf{J} dt, \quad (16b)$$

$$\mathbf{E} = \partial_t^2 \mathbf{A}_2 + \nabla\partial_t \phi_{A2} - \nabla\phi_A. \quad (16c)$$

Whereas the magnetic field is

$$\mathbf{H} = -\frac{1}{\mu\varepsilon}\nabla \times \nabla \times \mathbf{Ç}_2 + \frac{1}{\mu\varepsilon}\int \left(\nabla \times \mathbf{P}_{\text{ext}} + \int \nabla \times \mathbf{J} dt\right) dt - \frac{\mu_0}{\mu}\mathbf{M}_{\text{ext}}, \quad (16d)$$

$$\mathbf{H} = -\frac{1}{\mu}\nabla\times\partial_t\mathbf{A}_2 - \frac{\mu_0}{\mu}\mathbf{M}_{\text{ext}}, \tag{16e}$$

$$\mathbf{H} = \partial_t^2\mathbf{Ç}_2 + \partial_t\nabla\phi_{C2} - \nabla\phi_C. \tag{16f}$$

It is straight forward to recast these results in terms of the magnetic induction and electric displacement. The magnetic induction $\mathbf{B} = \mu\mathbf{H} + \mu_0\mathbf{M}_{\text{ext}}$, can be written as the curl of the potentials from (16e), $\mathbf{B} = \nabla\times\mathbf{A} = -\nabla\times\partial_t\mathbf{A}_2$. The electric displacement $\mathbf{D} = \varepsilon\mathbf{E} + \mathbf{P}_{\text{ext}}$, can also be written as the curl of a vector from (16b), $\mathbf{D} = -\nabla\times\mathbf{Ç} = \nabla\times\partial_t\mathbf{Ç}_2$ provided that the current $\mathbf{J}$ is zero.

The electromagnetic first order differential equations for the potpotentials are

$$\nabla\cdot\mathbf{A}_2 = \int\left(-\nabla^2\phi_{A2} + \int\left(\nabla^2\phi_A - \frac{1}{\varepsilon}\nabla\cdot\mathbf{P}_{\text{ext}} + \frac{\rho}{\varepsilon}\right)dt\right)dt. \tag{17a}$$

$$\nabla\cdot\mathbf{Ç}_2 = \int\left(-\nabla^2\phi_{C2} + \int\left(\nabla^2\phi_C - \frac{\mu_0}{\mu}\nabla\cdot\mathbf{M}_{\text{ext}}\right)dt\right)dt. \tag{17b}$$

$$\nabla\times\mathbf{Ç}_2 = \varepsilon\partial_t\mathbf{A}_2 + \varepsilon\nabla\phi_{A2} - \varepsilon\int\nabla\phi_A dt + \int\left(\mathbf{P}_{\text{ext}} + \int\mathbf{J}dt\right)dt. \tag{17c}$$

$$\nabla\times\mathbf{A}_2 = -\mu\partial_t\mathbf{Ç}_2 - \mu\nabla\phi_{C2} + \mu\int\nabla\phi_C dt - \mu_0\int\mathbf{M}_{\text{ext}}dt. \tag{17d}$$

The wavelike second order differential equations are obtained from the curl of (17d) and (17c),

$$\begin{aligned}\mu\varepsilon\nabla^2\mathbf{A}_2 - \mu\varepsilon\partial_t^2\mathbf{A}_2 = {}& \nabla\left(\nabla\cdot\mathbf{A}_2\right) + \mu\varepsilon\partial_t\nabla\phi_{A2} - \mu\varepsilon\nabla\phi_A \\ & + \mu\mathbf{P}_{\text{ext}} + \mu_0\nabla\times\int\mathbf{M}_{\text{ext}}dt + \mu\int\mathbf{J}dt,\end{aligned} \tag{18a}$$

$$\begin{aligned}\nabla^2\mathbf{Ç}_2 - \mu\varepsilon\partial_t^2\mathbf{Ç}_2 = {}& \nabla\left(\nabla\cdot\mathbf{Ç}_2\right) + \mu\varepsilon\partial_t\nabla\phi_{C2} - \mu\varepsilon\nabla\phi_C \\ & + \mu_0\mathbf{M}_{\text{ext}} - \int\left(\nabla\times\mathbf{P}_{\text{ext}} + \int\nabla\times\mathbf{J}dt\right)dt.\end{aligned} \tag{18b}$$

From (12a), it is clear that the six variables of the vector fields are being mapped into eight variables, two vectors with three components and two scalar fields. Due to this underdeterminacy, two additional conditions can be assigned [22]. Two interesting possibilities are envisaged:

If the magnetic vector and scalar potentials are set equal to zero, $\mathbf{Ç}_j = 0, \phi_C = 0$, we are left with the $\mathbf{A}_j, \phi_{Aj}$ potentials that are still susceptible of a gauge transformation (The non vanishing choice can of course be assigned to the magnetic potential). A highly appealing formulation in this scheme, is that a manifestly covariant four vector

$(\phi_{Aj}, \mathbf{A}_j)$ can be constructed. However, the highly symmetric tiered structure collapses because although the magnetic field $\mathbf{H}$ can still be written in terms of $\mathbf{A}$ via (12c), it cannot longer be written in terms of a $\mathbf{Ç}_j$. A similar argument holds for $\mathbf{E}$. Notice that four conditions are being imposed on the eight variables but as it is well known, the differential equations reduce to only two.

If the scalar potentials are set equal to zero, the underdeterminacy is lifted, since the six vector field components are mapped onto six vector potentials components. The potentials are then simply minus time derivatives of the fields. It is true that the fields can also be written in terms of the curl of the 'other' potential as a result of Maxwell's curl equations, but these expressions are now an alternative representation rather than definitions, i.e. (12b) and (12c) are a consequence of (12a). In the covariant formulation, setting the scalar potential equal to zero is recognized as the temporal gauge [23]. However, in the present scenario, it seems to be merely the removal of the redundancy of two extra variables. If this view is adopted, no gauge has been chosen by setting $\phi_{Aj} = \phi_{Cj} = 0$.

In the following two subsections, the conventional gauge approach is employed. The eight variables are retained but Lorenz like conditions are imposed on both potentials and potpotentials.

### 3.3. Potentials in the Lorenz gauge

The Lorenz gauge condition is chosen so that both potentials satisfy a wave equation,

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon\partial_t\phi_A, \qquad \nabla \cdot \mathbf{Ç} = -\mu\varepsilon\partial_t\phi_C. \tag{19}$$

The second order differential equations for the potentials in the Lorenz gauge are

$$\nabla^2\mathbf{A} - \mu\varepsilon\partial_t^2\mathbf{A} = -\mu\partial_t\mathbf{P}_{\text{ext}} - \mu_0\nabla\times\mathbf{M}_{\text{ext}} - \mu\mathbf{J}. \tag{20a}$$

$$\nabla^2\mathbf{Ç} - \mu\varepsilon\partial_t^2\mathbf{Ç} = -\mu_0\varepsilon\partial_t\mathbf{M}_{\text{ext}} + \nabla\times\mathbf{P}_{\text{ext}} + \int \nabla\times\mathbf{J}dt. \tag{20b}$$

$$\nabla^2\phi_A - \mu\varepsilon\partial_t^2\phi_A = \frac{1}{\varepsilon}\nabla\cdot\mathbf{P}_{\text{ext}} - \frac{\rho}{\varepsilon}. \tag{20c}$$

$$\nabla^2\phi_C - \mu\varepsilon\partial_t^2\phi_C = \frac{\mu_0}{\mu}\nabla\cdot\mathbf{M}_{\text{ext}}. \tag{20d}$$

### 3.4. Potpotentials in Lorenz gauge

In the vein of the Lorenz gauge, if $\mathbf{A}_2$ should satisfy an uncoupled wave equation, from (18a), the Lorenz condition for the potpotential is

$$\nabla\cdot\mathbf{A}_2 = -\mu\varepsilon\partial_t\phi_{A2} + \mu\varepsilon\phi_A. \tag{21a}$$

From the time derivative of (17a), and the time integral of the $\phi_A$ wave equation in the Lorenz gauge (20c), $\mu\varepsilon\partial_t\phi_A = \int\nabla^2\phi_A dt - \frac{1}{\varepsilon}\int\nabla\cdot\mathbf{P}_{\text{ext}}dt + \int\frac{\rho}{\varepsilon}dt$. Thus $\phi_{A2}$ satisfies a homogeneous wave equation even in the presence of sources. The Lorenz like condition for $\mathbf{Ç}_2$ is

$$\nabla \cdot \mathbf{Ç}_2 = -\mu\varepsilon\partial_t\phi_{C2} + \mu\varepsilon\phi_C. \tag{21b}$$

From (17b), recalling that in the Lorenz gauge $\phi_C$ satisfies (20d), a homogeneous wave equation for $\phi_{C2}$ is also obtained. The wave equations in the Lorenz gauge for the vector and scalar potentials with sources are

$$\nabla^2\mathbf{A}_2 - \mu\varepsilon\partial_t^2\mathbf{A}_2 = \mu\mathbf{P}_{\text{ext}} + \mu_0\nabla\times\int\mathbf{M}_{\text{ext}}\,dt + \mu\int\mathbf{J}dt \tag{22a}$$

$$\nabla^2\mathbf{Ç}_2 - \mu\varepsilon\partial_t^2\mathbf{Ç}_2 = \mu_0\mathbf{M}_{\text{ext}} - \int\left(\nabla\times\mathbf{P}_{\text{ext}}\,dt + \int\nabla\times\mathbf{J}\right)dt \tag{22b}$$

$$\nabla^2\phi_{A2} - \mu\varepsilon\partial_t^2\phi_{A2} = 0. \tag{22c}$$

$$\nabla^2\phi_{C2} - \mu\varepsilon\partial_t^2\phi_{C2} = 0. \tag{22d}$$

The electric field can be written in terms solely of the $\mathbf{A}_2$ potential, from (16c) and the Lorenz condition for $\mathbf{A}_2$,

$$\mathbf{E} = \partial_t^2\mathbf{A}_2 - \frac{1}{\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}_2\right). \tag{23a}$$

This result can also be obtained from the curl curl $\mathbf{A}_2$ equation (16a), invoking the $\nabla\times\nabla\times\mathbf{A}_2 = \nabla\left(\nabla\cdot\mathbf{A}_2\right) - \nabla^2\mathbf{A}_2$ vector identity and the $\mathbf{A}_2$ wave equation (22a). Similarly, the magnetic field from (16f) and (21b) is,

$$\mathbf{H} = \partial_t^2\mathbf{Ç}_2 - \frac{1}{\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{Ç}_2\right). \tag{23b}$$

## 4. Polarization potentials or Hertz vectors

The fields and potentials in terms of the Hertz electric polarization potential are defined as (constants differ depending on EM units) [18, 9, 14], $\phi_A = -\nabla\cdot\mathbf{\Pi}_e$, $\mathbf{A} = \mu\varepsilon\partial_t\mathbf{\Pi}_e$, $\mathbf{E} = \nabla\left(\nabla\cdot\mathbf{\Pi}_e\right) - \mu\varepsilon\partial_t^2\mathbf{\Pi}_e$, $\mathbf{B} = \mu\varepsilon\nabla\times\partial_t\mathbf{\Pi}_e$. The wave equation for the electric polarization potential is $\nabla^2\mathbf{\Pi}_e - \mu\varepsilon\partial_t^2\mathbf{\Pi}_e + \frac{1}{\varepsilon}\mathbf{P}_{\text{ext}} = 0$. Comparison with the present results, summarized in table 3, require

$$\phi_{A2} = 0, \quad -\frac{1}{\mu\varepsilon}\mathbf{A}_2 = \mathbf{\Pi}_e. \tag{24a}$$

The electric Hertz vector potential is then the second tier vector potpotential evaluated in the Lorenz gauge, scaled by a $-\frac{1}{\mu\varepsilon}$ factor. Notice that the vanishing scalar potpotential is consistent with (22c) even in the presence of a polarization source.

| | Eq. Num. |
|---|---|
| $\nabla \cdot \mathbf{A}_2 = -\mu\varepsilon\partial_t\phi_{A2} + \mu\varepsilon\phi_A$ | (21a) |
| $\mathbf{A} = -\partial_t \mathbf{A}_2 - \nabla\phi_{A2}$ | (15a) |
| $\mathbf{E} = \partial_t^2 \mathbf{A}_2 - \frac{1}{\mu\varepsilon}\nabla\left(\nabla \cdot \mathbf{A}_2\right)$ | (23a) |
| $\mathbf{B} = -\nabla \times \partial_t \mathbf{A}_2$ | from (16e) |

**Table 3.** Potpotential $\mathbf{A}_2$

| | Eq. |
|---|---|
| $\nabla \cdot \mathbf{Ç}_2 = -\mu\varepsilon\partial_t\phi_{C2} + \mu\varepsilon\phi_C$ | (21b) |
| $\mathbf{A} = -\frac{1}{\varepsilon}\nabla \times \mathbf{Ç}_2 - \int \nabla\phi_A dt + \frac{1}{\varepsilon}\int \left(\mathbf{P}_{\text{ext}} dt + \int \mathbf{J} dt\right) dt$ | (15b) |
| $\mathbf{B} = -\frac{1}{\varepsilon}\nabla \times \nabla \times \mathbf{Ç}_2 + \frac{1}{\varepsilon}\int \left(\nabla \times \mathbf{P}_{\text{ext}} + \int \nabla \times \mathbf{J} dt\right) dt$ | from (16d) |
| $\mathbf{E} = \frac{1}{\varepsilon}\nabla \times \partial_t \mathbf{Ç}_2 - \frac{1}{\varepsilon}\mathbf{P}_{\text{ext}} - \frac{1}{\varepsilon}\int \mathbf{J} dt$ | (16b) |

**Table 4.** Potpotential $\mathbf{Ç}_2$.

The Hertz magnetic potential is usually defined as $\phi_C = 0$, $\mathbf{A} = \nabla \times \mathbf{\Pi}_m$, $\mathbf{B} = \nabla \times \nabla \times \mathbf{\Pi}_m$, $\mathbf{E} = -\nabla \times \partial_t \mathbf{\Pi}_m$, with wave equation $\nabla^2 \mathbf{\Pi}_m - \mu\varepsilon\partial_t^2 \mathbf{\Pi}_m + \mu \mathbf{M} = 0$. Comparison of the dependence of these variables with the potpotential $\mathbf{Ç}_2$, sumarized in table 4, require that

$$\phi_{C2} = 0, \quad -\frac{1}{\varepsilon}\mathbf{Ç}_2 = \mathbf{\Pi}_m, \tag{24b}$$

provided that $\mathbf{P}_{\text{ext}}$ and $\mathbf{J}$ are zero. The role of the magnetic potential $\mathbf{\Pi}_m$, although similar, does not seem to be equivalent to the electric potential $\mathbf{\Pi}_e$. However, if the magnetic field $\mathbf{H}$ and the potential $\mathbf{Ç}$ are written in terms of $\mathbf{Ç}_2$, i.e. Eq. (23b) and (15a), they become fully symmetric. The dual Hertz vectors recently introduced by Elbistan [24] to describe the conservation of optical helicity, nicely fit into this scheme. The Elbistan potentials $\mathbf{Z}_A$ and $\mathbf{Z}_C$ are the second tier potentials $\mathbf{Ç}_2$ and $\mathbf{A}_2$ respectively (except for constant factors).

The polarization potentials were introduced separately, the electric polarization potential by Hertz [25] and the magnetic polarization potential by Righi [26]. However, they were afterwards lumped together [7, 17], and it has become customary to combine both polarization potentials [13]. The potential $\mathbf{A}$ is then defined in terms of the Hertz potentials as $\mathbf{A} = \mu\varepsilon\partial_t \mathbf{\Pi}_e + \nabla \times \mathbf{\Pi}_m$. Clearly, this definition no longer preserves the structure of Maxwell's equations as they do in the original Hertz and Righi definitions. Furthermore, the lumped proposal leads to rather unsymmetrical field definitions in terms of these 'Hertz potentials', $\mathbf{E} = \nabla(\nabla \cdot \mathbf{\Pi}_e) - \mu\varepsilon\partial_t^2 \mathbf{\Pi}_e - \nabla \times \partial_t \mathbf{\Pi}_m$ and $\mathbf{B} = \nabla \times \nabla \times \mathbf{\Pi}_m + \mu\varepsilon \nabla \times \partial_t \mathbf{\Pi}_e$ (constant coefficients differ for various authors). In their approach, the polarization potentials $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ are then treated as independent quantities [17]. For example, for an electric dipole source, the magnetic potential $\mathbf{\Pi}_m$ is set equal to zero. The motivation for grouping the two polarization potentials comes from another albeit related problem, the obtention of linearly independent vector solutions from a known vector solution.

### *4.1. Dipole radiation revisited*

Reconsider the original problem solved by Hertz, namely, the radiation of an electric dipole. The wave equations for the fields, potentials and potpotentials are abridged in table 5. From these, only four wave equations exhibit source terms that do not involve integral or differential operators. If the remaining greyed out terms are zero, the solutions can be readily given in terms of retarded Green functions.

The $\mathbf{A}_2$ potpotential wave equation without magnetic polarization nor free current is $\nabla^2 \mathbf{A}_2 - \mu\varepsilon\partial_t^2 \mathbf{A}_2 = \mu \mathbf{P}_{\text{ext}}$. The solution is the Green function $\mathbf{A}_2 = -\mu \int \frac{[\mathbf{P}_{\text{ext}}]}{R} dV$, where $R = \mathbf{r} - \mathbf{r}_0$, the position of the external electric polarization element is $\mathbf{r}_0$ and the square brackets represent retarded functions, $[f(t)] \equiv f\left(t - \frac{R}{c}\right)$. For an electric dipole with polarization $\mathbf{P}_{\text{ext}} = p(t)\,\delta(\mathbf{r} - \mathbf{r}_0)\,\hat{\mathbf{n}}$, $p(t)$ is the dipole moment with arbitrary time dependence, $\delta$ is a Dirac delta function and the unit vector $\hat{\mathbf{n}}$ specifies the direction of the dipole. The electric potpotential is then

$$\mathbf{A}_2 = -\mu \frac{[p]\,\hat{\mathbf{n}}}{R}. \tag{25a}$$

From the Lorenz conditions for the potential $\nabla \cdot \mathbf{A} = -\mu\varepsilon\partial_t \phi_A$ and the potpotential

| | | Eq. |
|---|---|---|
| $\nabla^2\mathbf{E}-\mu\varepsilon\partial_t^2\mathbf{E}=$ $\frac{1}{\varepsilon}\nabla\rho-\frac{1}{\varepsilon}\nabla\left(\nabla\cdot\mathbf{P}_{\mathrm{ext}}\right)$ $+\mu\partial_t^2\mathbf{P}_{\mathrm{ext}}+\mu_0\nabla\times\partial_t\mathbf{M}_{\mathrm{ext}}+\mu\partial_t\mathbf{J}$ | | (11a) |
| $\nabla^2\mathbf{H}-\mu\varepsilon\partial_t^2\mathbf{H}=$ $-\frac{\mu_0}{\mu}\nabla\left(\nabla\cdot\mathbf{M}_{\mathrm{ext}}\right)$ $+\mu_0\varepsilon\partial_t^2\mathbf{M}_{\mathrm{ext}}-\nabla\times\partial_t\mathbf{P}_{\mathrm{ext}}-\nabla\times\mathbf{J}$ | | (11b) |
| $\nabla^2\mathbf{A}-\mu\varepsilon\partial_t^2\mathbf{A}=-\mu\mathbf{J}$ $-\mu\partial_t\mathbf{P}_{\mathrm{ext}}-\mu_0\nabla\times\mathbf{M}_{\mathrm{ext}}$ | Point charge in motion | (20a) |
| $\nabla^2\phi_A-\mu\varepsilon\partial_t^2\phi_A=$ $-\frac{\rho}{\varepsilon}+\frac{1}{\varepsilon}\nabla\cdot\mathbf{P}_{\mathrm{ext}}$ | (Liénard-Wiechert) | (20c) |
| $\nabla^2\mathbf{Ç}-\mu\varepsilon\partial_t^2\mathbf{Ç}=$ $-\mu_0\varepsilon\partial_t\mathbf{M}_{\mathrm{ext}}+\nabla\times\mathbf{P}_{\mathrm{ext}}+\int\nabla\times\mathbf{J}dt$ | | (20b) |
| $\nabla^2\mathbf{A}_2-\mu\varepsilon\partial_t^2\mathbf{A}_2=$ $\mu\mathbf{P}_{\mathrm{ext}}$ $+\mu_0\int\nabla\times\mathbf{M}_{\mathrm{ext}}dt+\mu\int\mathbf{J}dt$ | Electric dipole (Hertz) | (22a) |
| $\nabla^2\mathbf{Ç}_2-\mu\varepsilon\partial_t^2\mathbf{Ç}_2=$ $\mu_0\mathbf{M}_{\mathrm{ext}}$ $-\int\left(\nabla\times\mathbf{P}_{\mathrm{ext}}dt+\int\nabla\times\mathbf{J}\right)d$ | Magnetic dipole (Righi) | (22d) |

**Table 5.** Wave equations useful for different configurations of the source terms. Solutions can be given in terms of the Green's function if there are no operators acting on the source terms (greyed out terms involve time or spatial operators).

$\nabla \cdot \mathbf{A}_2 = -\mu\varepsilon\partial_t\phi_{A2} + \mu\varepsilon\phi_A$, letting $\phi_{A2} = 0$, the scalar potential is

$$\phi_A = \frac{1}{\mu\varepsilon}\nabla \cdot \mathbf{A}_2 = \frac{1}{\varepsilon}\left(\frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3}\right)\mathbf{R} \cdot \hat{\mathbf{n}}. \tag{25b}$$

The usual way to evaluate the electric field with the Hertz potential $\mathbf{\Pi}_e = -\frac{1}{\mu\varepsilon}\mathbf{A}_2$, is from the second time derivative of the electric potpotential (23a). However, in the present formalism, the electric field can also be obtained from the $\mathbf{Ç}_2$ potpotential, (that would be set to zero if $\mathbf{A}$ were defined with the Hertz potentials lumped together). The wave equation for $\mathbf{Ç}_2$ (22b), does not have a simple solution for $\mathbf{P}_{\mathrm{ext}} \neq 0$. However, its solution can be obtained from $\mathbf{A}_2$ using (17d), setting the scalar magnetic potentials equal to zero $\phi_{C2} = \phi_C = 0$, then $\partial_t\mathbf{Ç}_2 = -\frac{1}{\mu}\nabla \times \mathbf{A}_2$. The magnetic potpotential is then

$$\mathbf{Ç}_2 = \left(\frac{1}{c}\frac{[p]}{R^2} + \frac{\int [p]\, dt}{R^3}\right)\hat{\mathbf{n}} \times \mathbf{R}. \tag{26a}$$

Notice that if $\mathbf{A}_2$ is a time dependent solenoidal field, by no means can $\mathbf{Ç}_2$ be zero. The magnetic field, from the second time derivative (16f), is

$$\mathbf{H} = \left(\frac{1}{c}\frac{[\ddot{p}]}{R^2} + \frac{[\dot{p}]}{R^3}\right)\hat{\mathbf{n}} \times \mathbf{R}. \tag{26b}$$

The electric field, obtained from the curl curl of the electric potpotential (16a), noting that $\nabla \times (\mathbf{R} \times \hat{\mathbf{n}}) = (\hat{\mathbf{n}} \cdot \nabla)\,\mathbf{R} - \hat{\mathbf{n}}\,(\nabla \cdot \mathbf{R}) = -2\hat{\mathbf{n}}$ and $\mathbf{R} \times (\mathbf{R} \times \hat{\mathbf{n}}) = (\mathbf{R} \cdot \hat{\mathbf{n}})\,\mathbf{R} - R^2\hat{\mathbf{n}}$, (details in appendix A) is

$$\mathbf{E} = \frac{1}{\varepsilon}\left(\frac{[\ddot{p}]}{c^2R^3} + \frac{3\,[\dot{p}]}{cR^4} + \frac{3\,[p]}{R^5}\right)(\mathbf{R} \cdot \hat{\mathbf{n}})\,\mathbf{R} - \frac{1}{\varepsilon}\left(\frac{[\ddot{p}]}{c^2R} + \frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3}\right)\hat{\mathbf{n}} - \frac{1}{\varepsilon}\mathbf{P}_{\mathrm{ext}}. \tag{26c}$$

This expression is equal to the usual result [18, p.86], except for the source term that is zero everywhere except at $\mathbf{r}_0$, where the dipole is located. An analogous procedure is followed to obtain the fields from magnetic dipole polarization sources, the starting point being the magnetic potpotential wave equation $\nabla^2\mathbf{Ç}_{2\mathrm{Md}} - \mu\varepsilon\partial_t^2\mathbf{Ç}_{2\mathrm{Md}} = \mu_0\mathbf{M}_{\mathrm{ext}}$.

Superposition is usually invoked for two or more solutions of the same linear differential equation. In the following case, superposition involves differential equations with two different source terms. Let $\mathbf{A}_{2\mathrm{Ed}}$ satisfy the electric dipole differential equation, $\nabla^2\mathbf{A}_{2\mathrm{Ed}} - \mu\varepsilon\partial_t^2\mathbf{A}_{2\mathrm{Ed}} = \mu\mathbf{P}_{\mathrm{ext}}$. The magnetic potpotential for a magnetic dipole source satisfies $\nabla^2\mathbf{Ç}_{2\mathrm{Md}} - \mu\varepsilon\partial_t^2\mathbf{Ç}_{2\mathrm{Md}} = \mu_0\mathbf{M}_{\mathrm{ext}}$. The electric potpotential for a magnetic dipole source is obtained from $\partial_t\mathbf{A}_{2\mathrm{Md}} = \frac{1}{\varepsilon}\nabla \times \mathbf{Ç}_{2\mathrm{Md}}$. This potpotential satisfies the differential equation $\nabla^2\mathbf{A}_{2\mathrm{Md}} - \mu\varepsilon\partial_t^2\mathbf{A}_{2\mathrm{Md}} = \mu_0\int \nabla \times \mathbf{M}_{\mathrm{ext}}\, dt$. The sum of the electric and magnetic dipoles differential equations for the potpotential $\mathbf{A}_2$ is

$$\nabla^2\left(\mathbf{A}_{2\mathrm{Ed}} + \mathbf{A}_{2\mathrm{Md}}\right) - \mu\varepsilon\partial_t^2\left(\mathbf{A}_{2\mathrm{Ed}} + \mathbf{A}_{2\mathrm{Md}}\right) = \mu\mathbf{P}_{\mathrm{ext}} + \mu_0\int \nabla \times \mathbf{M}_{\mathrm{ext}}\, dt.$$

A polarization source with electric and magnetic dipole moments then has solution $\mathbf{A}_2 = \mathbf{A}_{2\mathrm{Ed}} + \mathbf{A}_{2\mathrm{Md}}$. This superposition procedure is applicable to any of the fields or the potentials. Furthermore, it can also be extended to include charges and currents.

## 5. Vector wave equation solutions

For monochromatic fields in the absence of sources, the $\mathbf{A}_j$ (or $\mathbf{Ç}_j$) second order differential equations satisfy the vector Helmholtz equation

$$\nabla^2 \mathbf{A}_j + \mu\varepsilon\omega^2 \mathbf{A}_j = 0, \quad \nabla^2 \mathbf{Ç}_j + \mu\varepsilon\omega^2 \mathbf{Ç}_j = 0, \tag{27}$$

where $\omega$ is the angular frequency. The components of any of these vectors are unrelated unless the first order differential equations are taken into account. The div equations establish a relationship between the components of a given field and its direction of propagation, whereas the curl equations establish a relationship between the $\mathbf{A}_j$ and $\mathbf{Ç}_j$ components.

The Heaviside Larmor (HL) rotation symmetry (7), $\mathbf{A}_j \to \sqrt{\frac{\mu}{\varepsilon}}\mathbf{Ç}_j$, $\mathbf{Ç}_j \to -\sqrt{\frac{\varepsilon}{\mu}}\mathbf{A}_j$, leaves the electromagnetic equations invariant for any $j^{th}$ tier, these transformations are also solutions to the Helmholtz equation. A new vector solution $\mathbf{A}_j^{(g)}$ (or $\mathbf{Ç}_j^{(g)}$), can then be written as a linear combination of the particular solution and its HL symmetric solution,

$$\mathbf{A}_j^{(g)} = a_n \mathbf{A}_j + a_m \sqrt{\frac{\mu}{\varepsilon}}\mathbf{Ç}_j = a_n \mathbf{A}_j - a_m \frac{i}{k}\nabla \times \mathbf{A}_j, \tag{28a}$$

and

$$\mathbf{Ç}_j^{(g)} = a_n \mathbf{Ç}_j - a_m \sqrt{\frac{\varepsilon}{\mu}}\mathbf{A}_j = a_n \mathbf{Ç}_j - a_m \frac{i}{k}\nabla \times \mathbf{Ç}_j, \tag{28b}$$

where (5c), (5d) are used in the rightmost equalities and the wave vector magnitude is $k = \sqrt{\mu\varepsilon}\,\omega$. Since the divergence of $\mathbf{A}_j$ (or $\mathbf{Ç}_j$) is zero, a particular vector solution can be written as the curl of another vector, say $\mathbf{U}$, $\mathbf{A}_j^{(p)} = \frac{i}{k}\nabla \times \mathbf{U}$, the $k^{-1}$ inverse wave number insures the same units for $\mathbf{A}_j$ and $\mathbf{U}$. The imaginary unit is a convenient constant phase factor as will be seen here below. A relevant point being that the seed vector $\mathbf{U}$ can be, but does not need to be solenoidal. That is, $\mathbf{U}$ need not be in the $\mathbb{EM}$ set. Substitution in (27) gives, $\nabla \times \left(\nabla^2 \mathbf{U} + k^2 \mathbf{U}\right) = 0$. Therefore $\mathbf{U}$ must satisfy the inhomogeneous vector Helmholtz equation,

$$\nabla^2 \mathbf{U} + k^2 \mathbf{U} = \nabla \Upsilon. \tag{29}$$

To the best of our knowledge, only the case when $\nabla\Upsilon = 0$ has been previously before. In this case, $\mathbf{U}$ is a solution to the homogeneous vector wave equation. Let the general solution to the scalar wave equation be $U$; The directional property can be obtained from multiplication with a vector $\boldsymbol{a}$, such that $\nabla^2(\psi \boldsymbol{a}) = \boldsymbol{a}\nabla^2\psi$, this condition is satisfied if $\psi\nabla^2\boldsymbol{a} = -2(\nabla\psi \cdot \nabla)\boldsymbol{a}$. In particular, for any constant vector $\boldsymbol{a}$. The fields or the potentials in any tier can then be derived from a scalar solution of the Helmholtz equation from

$$\mathbf{A}_j^{(g)} = a_n \frac{i}{k}\nabla \times \mathbf{U} + a_m \frac{1}{k^2}\nabla \times \nabla \times \mathbf{U}, \tag{30a}$$

$$\mathbf{Ç}_j^{(g)} = \sqrt{\frac{\varepsilon}{\mu}}\left(-a_m \frac{i}{k}\nabla \times \mathbf{U} + a_n \frac{1}{k^2}\nabla \times \nabla \times \mathbf{U}\right). \tag{30b}$$

The Debye potentials are a particular case of this scheme if the problem exhibits spherical symmetry. Two scalar functions $u, v$, called the Debye potentials satisfy Helmholtz equation. The product of these functions times a unit radial vector $\mathbf{r}$, satisfy equations of the form (30a), (30b) with the identifications $\frac{a_m}{k^2}\mathbf{U} = u\mathbf{r}$ and $i\frac{a_n}{k^2}\mathbf{U} = v\mathbf{r}$. In this way, solutions to the vector wave equations satisfying Maxwell equations are obtained in spherical domains. The multipole expansion has been shown to be equivalent to the Debye formalism, through a series expansion of the Debye potentials [27]. It has been subsequently shown, that only a single Debye potential is actually required [28], as expected from the present derivation.

If the $\nabla\times\nabla\times\mathbf{U} = \nabla(\nabla\cdot\mathbf{U}) - \nabla^2\mathbf{U}$ identity is used and the wave equation invoked,

$$\mathbf{A}_j^{(g)} = a_m\mathbf{U} + a_m \frac{1}{k^2}\nabla(\nabla\cdot\mathbf{U}) + a_n\frac{i}{k}\nabla\times\mathbf{U}, \tag{31a}$$

$$\mathbf{Ç}_j^{(g)} = \sqrt{\frac{\varepsilon}{\mu}}\left(a_n\mathbf{U} + a_n\frac{1}{k^2}\nabla(\nabla\cdot\mathbf{U}) - a_m\frac{i}{k}\nabla\times\mathbf{U}\right). \tag{31b}$$

If $\mathbf{U}$ satisfies the free EM equations, then $\mathbf{U}$ is solenoidal, $\nabla\cdot\mathbf{U} = 0$. Thus $\mathbf{A}_j = a_m\mathbf{U} + \frac{i}{k}a_n\nabla\times\mathbf{U}$, provided that $\nabla\Upsilon = 0$. If $\mathbf{U} = \psi\boldsymbol{a}$ is a single component vector, $\mathbf{A}_0 = \mathbf{E} = a_n\frac{i}{k}\nabla\times\mathbf{U}$ (or $\mathbf{Ç}_0 = \mathbf{H} = -\sqrt{\frac{\varepsilon}{\mu}}a_m\nabla\times\mathbf{U}$) has two non vanishing components and is thus well suited to account for TE (or TM) modes. This result has long been used to describe the fields within wave-guides and more recently to model propagation invariant optical vector fields [29]. This procedure can be extended to a non solenoidal potential $\mathbf{A}$ if it satisfies the Lorenz condition, because (27) still holds. An additional gradient of a function is nonetheless required in (30a). Since $\nabla\cdot\mathbf{A}_1 = -\mu\varepsilon\partial_t\phi_A$ in the Lorenz gauge, the $\mathbf{A}_1$ potential given by (30a) for $j = 1$, has to be modified to $\mathbf{A}_1 = -\frac{i}{\omega}\nabla\phi_A + a_n\frac{i}{k}\nabla\times\mathbf{U} + a_m\frac{1}{k^2}\nabla\times\nabla\times\mathbf{U}$.

The similarity of (30a)-(30b) with the expressions for the fields in terms of the potpotentials (Table 1), were the motivation that led to the identification [7, p.394] $\mathbf{\Pi}_e \dashrightarrow a_n\mathbf{U}$, $\mathbf{\Pi}_m \dashrightarrow a_m\mathbf{U}$. Due to the arbitrariness of the $a_m, a_n$ coefficients, the electric and magnetic potentials defined in this way, no longer bear an unequivocal relationship between them. This identification is not correct if the Hertz polarization potentials correspond to the potpotentials (24a) and (24b). In fact, notice that from (30a)-(30b), the polarization potentials themselves can be written in terms of a particular vector solution $\mathbf{U}$, for example, $-\mu\varepsilon\mathbf{\Pi}_e = \mathbf{A}_2 = a_n\frac{i}{k}\nabla\times\mathbf{U} + a_m\frac{1}{k^2}\nabla\times\nabla\times\mathbf{U}$. The procedure followed here has been somewhat similar to previous derivations [7, p.393]. However, the two vectors (labeled $\mathbf{M}$ and $\mathbf{N}$ in other texts), are not only the curl of each other but involve a temporal derivative i.e. (5c), (5d). For monochromatic waves in the complex formalism, the time derivative involves the imaginary unit. Careful book keeping of the $i$'s has been made, this caution maintains the appropriate polarization transformation properties.

Another approach in order to obtain vector solutions has been to consider a particular single component solution for $\mathbf{A}$. The scalar potential is then obtained from the Lorenz condition and thereafter the electric field is evaluated [10]. This formulation was followed by Allen et al. [11, 30] to establish the orbital angular momentum of Laguerre-Gaussian modes. However, this scheme cannot be made entirely consistent for the electromagnetic fields and their corresponding potentials but in the lowest order. This can be seen as follows: If $\mathbf{A}$ has only one component, then $\mathbf{B}$ has at most two non zero components. Due to the finite transverse spatial extent of the beams, Maxwell's equations couple their polarization and spatial degrees of freedom. For this reason, in general, all three Cartesian components of the fields should be nonzero even in free space propagation [31]. In contrast, the solutions (31a)-(31b) generate vector fields with the three non vanishing components for both, the electric and magnetic fields. If $U$ is solution to the paraxial wave equation instead of the wave equation, the vector expressions will be correct within the paraxial approximation.

## 6. Conclusions

The fields and potentials in any tier have been presented in a unified form. To this end, it has been crucial to recover the $\mathbf{Ç}$ potential from oblivion [4]. In fact, it actually came back to the fore of electromagnetic theory only in the last decade, so that the optical helicity density and its flow could satisfy the Heaviside Larmor symmetry [32, 1, 3, 5], an exception being the early paper of Afanasiev and Stepanovsky [33]. In order to exhibit the full symmetry of the equations, it is necessary to use either the fields $\mathbf{E}, \mathbf{H}$ or the electric displacement and magnetic induction $\mathbf{D}, \mathbf{B}$ as the starting point. If the fields are monochromatic, the $\mathbf{A}_j$, $\mathbf{Ç}_j$ vector functions are repeated every second tier. The $\mathbf{A}_j$ potentials play the role of the electric field in the electromagnetic equations whereas the $\mathbf{Ç}_j$ potentials play the role of the magnetic field. It thus seems more appropriate to refer to $\mathbf{A}$ as the first tier electric potential rather than the magnetic vector potential as is usually done. In vacuum, the symmetry of $\mathbf{E}$ and $\mathbf{H}$ is almost complete, the exception being the minus sign required in the $\mathbf{H}$ to $\mathbf{E}$, dual symmetry transformation. The minus sign in the curl $\mathbf{E}$, Maxwell-Faraday equation being another facet of the same asymmetry. Physical reality is ascribed to the electric and magnetic fields. However, any of the potentials or the hyperpotentials could represent physical quantities equally well if these quantities are seen as time derivatives or time integrals of the fields, i.e. Eq. (1). This would be equivalent, if the analogue is permitted, to the physical reality of velocity, acceleration or higher order derivatives in a mechanical system.

However, this state of affairs is modified considerably when sources are present. The HL symmetry is broken for monopoles, although as evinced in (22a) and (22b), it retains its symmetry for dipoles if $\mathbf{P}_{\text{ext}}$ and $\mu_0\mathbf{M}_{\text{ext}}$ are also exchanged. The source terms single out the $\mathbf{E}, \mathbf{H}$ (or $\mathbf{A}_0$, $\mathbf{Ç}_0$) fields tier, where the source of the electric field is the electric charge, as stated in the $\nabla \cdot \mathbf{E} = \rho/\varepsilon$ equation. However, it is arguably that since $\rho$ and $\mathbf{J}$ are the source terms in the scalar and vector potentials wave equations, then $\phi_A$ and $\mathbf{A}$ should have physical reality. This view is reinforced by the interaction Hamiltonian function that places on an equal footing the charged particle momentum and the $\mathbf{A}$ vector potential, $\mathcal{H} = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2$. The $\mathbf{A}$ and $\mathbf{Ç}$ potentials are also present in the angular momentum of electromagnetic fields, where the helicity density flow is $\mathbf{\Phi}_{\mathbf{AÇ}} = \frac{\varepsilon}{2}\mathbf{E} \times \mathbf{A} + \frac{\mu}{2}\mathbf{H} \times \mathbf{Ç}$. The objection that observables should be independent of gauge transformations has been addressed in two different ways: i) angular momentum definitions have been restricted to transverse potential components [34, 35]. ii) the helicity and its flow have been redefined including their respective scalar potentials, $\mathbf{\Phi}_{\mathbf{AÇ}} = \frac{\varepsilon}{2}\left(\mathbf{E} - \nabla\phi_A\right) \times \mathbf{A} + \frac{\mu}{2}\left(\mathbf{H} - \nabla\phi_C\right) \times \mathbf{Ç}$, in such a way that the helicity conservation equation is gauge invariant [22].

The electromagnetic equations with sources have been written for any tier without specifying a gauge (17a)-(17d). From these expressions, any gauge can be evaluated. Three cases have been considered, i) $\mathbf{Ç}_j = 0, \phi_{Cj} = 0$, that destroys the tiered structure; ii) $\phi_{Aj} = 0, \phi_{Cj} = 0$, that can be seen as the removal of two redundant variables, leaving no room for gauge transformations. This zero scalar potentials condition can also be seen as the temporal gauge condition; iii) A Lorenz gauge extended to the two potentials $\mathbf{A}$ and $\mathbf{Ç}$ as well as the potpotentials $\mathbf{A}_2$ and $\mathbf{Ç}_2$. The Lorenz gauge has been worked out explicitly to place into context the four wave equations that exhibit the charge, current and polarization sources (Table 5). The Hertz potentials have been shown to be the potpotentials or second tier potentials in this scheme. We have argued in favour of defining the fields keeping the Hertz potentials separate. The solution of an electric dipole distribution with arbitrary time dependence has been performed emphasizing the previously unreported procedure that involves the $\mathbf{Ç}_2$ magnetic potpotential (detailed derivation in the supplementary file). The final outcome is of course the same except for the inclusion of the source term in the field's solution.

The obtention of vector solutions to the Helmholtz equation from a scalar solution, has been shown to be a direct consequence of linear superposition, the HL symmetry and the relationship between $\mathbf{A}_j$'s and $\mathbf{Ç}_j$'s via the curl electromagnetic equations. The Debye potentials have been shown to be the solutions to the spherical case. The fact that the div equations are not invoked allows for a scalar solution to be introduced even if the divergence of its product with a constant vector is not zero in the absence of sources. The form of the solutions (30a),(30b) can be used to obtain vector solutions to any of the fields or potentials. This form is a consequence of the tiered structure of the electromagnetic equations and should not be confused with the Hertz vector potentials. The two vectors involved are indeed related by their curls but a time derivative is also required in order to retain the correct polarization of the superposed solutions.

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## Appendix A. Dipole fields from potpotential $\mathbf{Ç}_2$

The electric dipole potpotential solution to the differential equation $\nabla^2\mathbf{A}_2 - \mu\varepsilon\partial_t^2\mathbf{A}_2 = \mu\mathbf{P}_{\rm ext}$, is

$$\mathbf{A}_2 = -\mu\frac{[p]\,\hat{\mathbf{n}}}{R}. \tag{A1}$$

The gradient of the potpotential vector coefficient is

$$\nabla\left(\frac{[p]}{R}\right) = -\left(\frac{1}{c}\frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3}\right)\mathbf{R}, \tag{A2}$$

and in turn, the gradient of the resulting coefficient is

$$\nabla\left(\frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3}\right) = -\left(\frac{[\ddot{p}]}{c^2R^3} + \frac{3\,[\dot{p}]}{cR^4} + 3\frac{[p]}{R^5}\right)\mathbf{R}. \tag{A3}$$

The divergence of the electric potpotential is

$$\nabla \cdot \mathbf{A}_2 = \mu \left( \frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3} \right) \mathbf{R} \cdot \hat{\mathbf{n}}. \tag{A4}$$

The curl of $\mathbf{A}_2$ is

$$\nabla \times \mathbf{A}_2 = \mu \left( \frac{1}{c} \frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3} \right) \mathbf{R} \times \hat{\mathbf{n}}. \tag{A5}$$

The magnetic potpotential is obtained from $\mathbf{A}_2$ using $\nabla \times \mathbf{A}_2 = -\mu \partial_t \boldsymbol{\c{C}}_2 - \mu \nabla \phi_{C2} + \mu \int \nabla \phi_C dt - \mu_0 \int \mathbf{M}_{\text{ext}} dt$, since $\mathbf{M}_{\text{ext}} = 0$ for an electric dipole and setting the scalar magnetic potentials equal to zero $\phi_{C2} = \phi_C = 0$,

$$\boldsymbol{\c{C}}_2 = -\frac{1}{\mu} \int \nabla \times \mathbf{A}_2 dt = \left( \frac{1}{c} \frac{[p]}{R^2} + \frac{\int [p]\, dt}{R^3} \right) \hat{\mathbf{n}} \times \mathbf{R}. \tag{A6}$$

The magnetic field, from the second time derivative of the magnetic potpotential is

$$\mathbf{H} = \partial_t^2 \boldsymbol{\c{C}}_2 = \left( \frac{1}{c} \frac{[\ddot{p}]}{R^2} + \frac{[\dot{p}]}{R^3} \right) \hat{\mathbf{n}} \times \mathbf{R}.$$

The electric field can be obtained from the magnetic potpotential (16b), $\mathbf{E} = \frac{1}{\varepsilon} \nabla \times \partial_t \boldsymbol{\c{C}}_2 - \frac{1}{\varepsilon} \mathbf{P}_{\text{ext}}$, or from the curl curl of the electric potpotential (16a), $\mathbf{E} = -\frac{1}{\mu\varepsilon} \nabla \times \nabla \times \mathbf{A}_2 - \frac{1}{\varepsilon} \mathbf{P}_{\text{ext}}$.

$$\nabla \times \partial_t \boldsymbol{\c{C}}_2 = -\frac{1}{\mu} \nabla \times \nabla \times \mathbf{A}_2 = -\nabla \left( \frac{1}{c} \frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3} \right) \times (\mathbf{R} \times \hat{\mathbf{n}}) - \left( \frac{1}{c} \frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3} \right) \nabla \times (\mathbf{R} \times \hat{\mathbf{n}})$$

From (A3) and noting that $\nabla \times (\mathbf{R} \times \hat{\mathbf{n}}) = (\hat{\mathbf{n}} \cdot \nabla) \mathbf{R} - \hat{\mathbf{n}} (\nabla \cdot \mathbf{R}) = -2\hat{\mathbf{n}}$, then

$$\nabla \times \partial_t \boldsymbol{\c{C}}_2 = \left( \frac{[\ddot{p}]}{c^2 R^3} + \frac{3\, [\dot{p}]}{cR^4} + \frac{3\, [p]}{R^5} \right) \mathbf{R} \times (\mathbf{R} \times \hat{\mathbf{n}}) - \left( \frac{1}{c} \frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3} \right) (-2\hat{\mathbf{n}})$$

but $\mathbf{R} \times (\mathbf{R} \times \hat{\mathbf{n}}) = (\mathbf{R} \cdot \hat{\mathbf{n}}) \mathbf{R} - R^2 \hat{\mathbf{n}}$, thus

$$\nabla \times \partial_t \boldsymbol{\c{C}}_2 = \left( \frac{[\ddot{p}]}{c^2 R^3} + \frac{3\, [\dot{p}]}{cR^4} + \frac{3\, [p]}{R^5} \right) \left( (\hat{\mathbf{n}} \cdot \mathbf{R}) \mathbf{R} - R^2 \hat{\mathbf{n}} \right) + \left( \frac{1}{c} \frac{[\dot{p}]}{R^2} + \frac{[p]}{R^3} \right) 2\hat{\mathbf{n}}$$

Grouping terms, the electric field is then

$$\begin{aligned} \mathbf{E} = \frac{1}{\varepsilon} \nabla \times \partial_t \boldsymbol{\c{C}}_2 - \frac{1}{\varepsilon} \mathbf{P}_{\text{ext}} = \frac{1}{\varepsilon} \left( \frac{[\ddot{p}]}{c^2 R^3} + \frac{3\, [\dot{p}]}{cR^4} + \frac{3\, [p]}{R^5} \right) (\hat{\mathbf{n}} \cdot \mathbf{R}) \mathbf{R} \\ - \frac{1}{\varepsilon} \left( \frac{[\ddot{p}]}{c^2 R} + \frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3} \right) \hat{\mathbf{n}} - \frac{1}{\varepsilon} \mathbf{P}_{\text{ext}}. \end{aligned} \tag{A7}$$

The electric field, evaluated in the usual way from the second time derivative of the potpotential recalling its relationship with the Hertz electric potential $\boldsymbol{\Pi}_e = -\frac{1}{\mu\varepsilon} \mathbf{A}_2$

[18, p.87] is

$$\mathbf{E} = \partial_t^2 \mathbf{A}_2 - \frac{1}{\mu\varepsilon}\nabla\left(\nabla\cdot\mathbf{A}_2\right) = \text{Ç}_2 \frac{1}{\varepsilon}\left(\frac{[\ddot{p}]}{c^2R^3} + \frac{3\,[\dot{p}]}{cR^4} + \frac{3\,[p]}{R^5}\right)\mathbf{R}\left(\mathbf{R}\cdot\hat{\mathbf{n}}\right) - \frac{1}{\varepsilon}\left(\frac{[\ddot{p}]}{c^2R} + \frac{[\dot{p}]}{cR^2} + \frac{[p]}{R^3}\right)\hat{\mathbf{n}}. \quad \text{(A8)}$$

Both results, as expected, are identical outside the source. At the source coordinates, the Green function solution diverges and is not valid. At the point $\mathbf{r}_0$, the electric field should be equal to the external polarization source, $\mathbf{E}\left(\mathbf{r}_0\right) = -\frac{1}{\varepsilon}\mathbf{P}_{\text{ext}}\left(\mathbf{r}_0\right)$, consistent with (A7).

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