To appear in the *Journal of Modern Optics*
Vol. 00, No. 00, 00 Month 20XX, 1–10

Helicity continuity equation for electromagnetic fields with sources.

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(*v4.0 released November 2013*)

The helicity continuity equation is derived from the wave equations of the electromagnetic potentials following the rationale of the complementary fields approach. The conserved quantity and its corresponding flow naturally arise from the conservation equation. The continuity equation is obtained for fields either in vacuum or homogeneous non-dispersive media in the presence of charges and/or currents. The derivation is otherwise quite general, there is no need to assume monochromatic fields nor a paraxial approximation. The symmetry of the electric and magnetic contributions is a consequence of the conserved quantity structure rather than an ad hoc hypothesis. The locally conserved quantities hold exactly without any averaging over time or space. This result is a hallmark of the complementary fields framework, whereby the energy content of the fields is dynamically exchanged between them.

Keywords: Angular momentum; Electromagnetic fields; Quantum mechanics; Helicity; Spin; Conservation equations.

1. Introduction

The helicity of light is important in the coupling between electromagnetic fields and matter, in particular, materials containing chiral structures [1]. Through the adequate control of the helicity, it is possible that molecular enantiomers relevant to biological and pharmaceutical studies could be measured and separated [2]. The helicity is a central concept in different fields such as magento-hydrodynamics, meteorology or particle physics. In hydrodynamics, the helicity provides a measure of the degree of linkage of vortex lines [3] and is conserved in smooth fields [4]. In field quantized theories, the helicity flow is associated with spin. At a fundamental level, a major controversy has been whether it is possible to separate the total angular momentum into spin and orbital parts in a gauge invariant way [5]. This issue is crucial to the measurement of these physical quantities in gauge theories like quantum electrodynamics and quantum chromodynamics.

Electromagnetic fields are capable of carrying angular momentum (AM) in addition to linear momentum. It is widely accepted, although alternatives exist [6, App. C], that the linear kinetic momentum density of a free electromagnetic field is described by Poynting's vector ($\mathbf{E} \times \mathbf{H}$). The AM density $\boldsymbol{J}_{mech} = \mathbf{r} \times \mathbf{p}$ obtained from these expressions is $\boldsymbol{J} = \mathbf{r} \times (\mathbf{E} \times \mathbf{H})$. Due to the cross product, the direction of this AM must be perpendicular to the direction of propagation. In contradiction with this result, circularly polarized waves are expected to have an AM parallel to the direction of propagation. Furthermore, there is ample experimental evidence of AM transfer in the wave vector direction from a circularly polarized field onto small particles [7, 8]. A crucial problem with this theoretical approach is where to locate the origin, in particular, in the all important case of plane waves. Some very clever schemes have been devised to establish an origin. For example, one proposal evaluated the AM from the central lobe produced by the superposition of four waves. The centre of the lobe

*Corresponding author. Email: mfg@xanum.uam.mx URL: http://luz.izt.uam.mx fixed an origin, thereafter the lobe dimensions were extended to infinity [9]. However, the standard workaround has been to consider the total AM $\boldsymbol{J}_{tot} = \int \mathbf{r} \times (\mathbf{E} \times \mathbf{H})\, dV$, instead of its density. Through a judicious use of vector identities, partial integration and dismissal of surface terms at infinity, $\boldsymbol{J}_{tot}$ can be written as the sum of two terms, $\boldsymbol{J}_{tot} = \int (\mathbf{E} \times \mathbf{A})\, dV + \int \sum_i \mathrm{E}_i\, (\mathbf{r} \times \nabla)\, \mathrm{A}_i dV$ [10]. These two terms have been associated with 'spin' and 'orbital' AM respectively. However, whether this separation is physically plausible has been strongly debated in recent years. This procedure is also invoked in quantum mechanics [11, p.46] in order to establish the photon AM [12, p.327]. A further subtlety is whether, the linear momentum is derived from the kinetic momentum $\mathbf{p} = m\dot{\mathbf{x}}$, or from the canonical momentum $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$ [5]. The different terms that comprise the total AM have also been catalogued as intrinsic or extrinsic [13]. The former depend only on properties of the field whereas the latter involve external parameters such as the location of the origin. The present communication is exclusively devoted to intrinsic AM, usually associated with the so called 'spin' AM term.

The notion of helicity for the free electromagnetic field was introduced following the static magnetic field topological invariant $\int (\mathbf{A} \cdot \mathbf{H})\, dV$, [14]. However, these quantities were shown not to satisfy the Heaviside-Larmor symmetry ($\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$) and more generally, duality transformations. Furthermore, it was later shown that these asymmetric electric-magnetic expressions were not conserved [15]. To remedy these problems, it was ad hoc proposed that the expressions had to be made symmetrical in the electric and magnetic fields ('electric–magnetic democracy') [6]. The differential forms of the integral expressions have become commonplace although the original derivation requires integral non local forms to dismiss surface integral expressions at infinity. Conservation type equations of the differential forms for free fields have been constructed by considering the sum of appropriate terms and showing that they add up to zero [16–18]. The conserved quantities have been associated with symmetry properties in accordance with Noether's theorem [19]. In particular, the rotational symmetry of the field vectors has been associated with 'spin' conservation [20].

In this paper, the helicity continuity equation is derived from Maxwell electromagnetic equations in the presence of charges and currents. The conserved quantity and its corresponding flow stem from the conservation equation rather than the other way around. The symmetry of the electric and magnetic contributions is a consequence of the derivation rather than an ad hoc hypothesis. The derivation does not invoke neither the linear nor the angular mechanical momentum, thus it does not rely on the kinetic or canonical versions of these quantities. Unlike Poynting's theorem, the helicity continuity equation does not require any averaging over time or space. It will be shown that in the presence of charge, it is only possible to establish a continuity equation solely in terms of the transverse vector potentials in the radiation zone.

2. Vector potential equations with sources

In the absence of charges or currents, the electromagnetic vector potential $\mathbf{A}$ satisfies the homogeneous wave equation either in the Lorenz or the Coulomb gauge. The scalar potential ϕ_A can be chosen equal to zero because two vector potential fields will be invoked to describe the EM fields. This condition is not essential for the derivation but it simplifies the procedure. The fields in terms of the potential are then given by $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial_t \mathbf{A}$. It is possible to introduce a second vector potential $\mathbf{C}$ [21, 22], such that in SI units, $\mathbf{B} = -\partial_t \mathbf{C}$ and $\mathbf{E} = -\frac{1}{\mu\varepsilon} \nabla \times \mathbf{C}$. In a highly symmetrical fashion with its $\mathbf{A}$ counterpart, the time derivative of the vector potential $\mathbf{C}$ is now related to the magnetic field whereas its curl is proportional to the electric field. Again, the vector potential $\mathbf{C}$ satisfies a wave equation. The two vector potentials are not independent, from Maxwell-Faraday's induction equation, $\nabla \times \partial_t \mathbf{A} = -\partial_t^2 \mathbf{C}$. In the presence of charges $\nabla \cdot \mathbf{E} \neq 0$, thus the relationship $\mathbf{E} = -\frac{1}{\mu\varepsilon} \nabla \times \mathbf{C}$ has to be modified to allow for a non zero divergence. Let

$$\mathbf{E} = -\frac{1}{\mu\varepsilon} \nabla \times \mathbf{C} + f = -\partial_t \mathbf{A}, \tag{1}$$

where f is a function yet to be determined. From the divergence of this equation and the first of Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \nabla \cdot f = \frac{\rho}{\varepsilon} = -\partial_t \nabla \cdot \mathbf{A}.$$

Substitution of the fields in terms of the time derivatives of the vector potentials in the Maxwell-Ampère equation $\nabla \times \mathbf{B} = \mu\varepsilon\partial_t \mathbf{E} + \mu \mathbf{J}$ and subsequent temporal integration of all terms, gives $\nabla \times \mathbf{C} = \mu\varepsilon\partial_t \mathbf{A} - \mu \int \mathbf{J} dt$. The function f is obtained from comparison with (1), $f = -\frac{1}{\varepsilon} \int \mathbf{J} dt$. The new set of equations for the two vector potentials with sources are

$$\nabla \cdot \mathbf{A} = -\int \frac{\rho}{\varepsilon} dt, \tag{2}$$

$$\nabla \cdot \mathbf{C} = 0, \tag{3}$$

$$\nabla \times \mathbf{C} = \mu\varepsilon\partial_t \mathbf{A} - \mu \int \mathbf{J} dt, \tag{4}$$

$$\nabla \times \mathbf{A} = -\, \partial_t \mathbf{C}. \tag{5}$$

Time integration constants have been set to zero throughout. From the time derivative of these equations, Maxwell's equations for the electromagnetic fields are readily obtained. The electromagnetic fields in terms of the two vector potentials are then

$$\mathbf{E} = -\frac{1}{\mu\varepsilon} \nabla \times \mathbf{C} - \frac{1}{\varepsilon} \int \mathbf{J} dt = -\partial_t \mathbf{A}, \tag{6}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = -\partial_t \mathbf{C}. \tag{7}$$

Form the divergence of (6), $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$ is obtained using the conservation of charge. These identifications are not unique, other attempts to include sources have been recently proposed [22–24]. An asset of the present assignment is that it produces a replica of Maxwell's equations with the corresponding time integrated sources. Furthermore, this proposal is not restricted to the transverse component of the current density, since the divergence of Eq. (6) does not impose such condition. Evaluation of the curl of equations (4) and (5) lead to the inhomogeneous wave equations for the vector potentials

$$\nabla^2 \mathbf{C} - \mu\varepsilon\, \partial_t^2 \mathbf{C} = \mu \int \nabla \times \mathbf{J}\, dt, \tag{8}$$

$$\nabla^2 \mathbf{A} - \mu\varepsilon\, \partial_t^2 \mathbf{A} = -\frac{1}{\varepsilon} \int \nabla \rho\, dt - \mu\, \mathbf{J}. \tag{9}$$

3. Complementary fields

A continuity equation may be obtained from the scalar wave equation with the complementary fields formulation [25]. This procedure is now extended to vector fields. The vector potentials $\mathbf{A}$ and $\mathbf{C}$

are now the two complementary field functions. The fields are complementary in the sense that the energy content of the system is continuously transferred back and forth between these two fields. Evaluation of the inner product of the vector potential $\mathbf{A}$ with the wave equation (8) for the vector potential $\mathbf{C}$ gives,

$$\mathbf{A}\cdot\nabla^2\mathbf{C}-\mu\varepsilon\mathbf{A}\cdot\partial_t^2\mathbf{C}=\mu\mathbf{A}\cdot\int\nabla\times\mathbf{J}dt.$$

Evaluation of the inner product of the vector potential $\mathbf{C}$ with the wave equation (9) for the vector potential $\mathbf{A}$ produces,

$$\mathbf{C}\cdot\nabla^2\mathbf{A}-\mu\varepsilon\mathbf{C}\cdot\partial_t^2\mathbf{A}=-\frac{1}{\varepsilon}\mathbf{C}\cdot\int\nabla\rho dt-\mu\mathbf{C}\cdot\mathbf{J}.$$

Subtracting these two equations

$$\begin{aligned}\left(\mathbf{A}\cdot\nabla^2\mathbf{C}-\mathbf{C}\cdot\nabla^2\mathbf{A}\right)+\mu\varepsilon\left(\mathbf{C}\cdot\partial_t^2\mathbf{A}-\mathbf{A}\cdot\partial_t^2\mathbf{C}\right)=\\ \frac{1}{\varepsilon}\mathbf{C}\cdot\int\nabla\rho dt+\mu\mathbf{C}\cdot\mathbf{J}+\mu\mathbf{A}\cdot\int\nabla\times\mathbf{J}dt.\end{aligned}\tag{10}$$

Which equation is subtracted from the other is a matter of convention, the choice has been made so that right hand circularly polarized light propagating in the positive direction has negative helicity. The first term involving second order spatial derivatives on the left-hand side of (10), can be rewritten in terms of a divergence with the aid of a vector version of Green's second identity [26],

$$\left(\mathbf{A}\cdot\nabla^2\mathbf{C}-\mathbf{C}\cdot\nabla^2\mathbf{A}\right)=\nabla\cdot\left[\mathbf{A}\left(\nabla\cdot\mathbf{C}\right)-\mathbf{C}\left(\nabla\cdot\mathbf{A}\right)+\mathbf{A}\times\nabla\times\mathbf{C}-\mathbf{C}\times\nabla\times\mathbf{A}\right].$$

Invoking (2) and (3), the first two terms within the divergence are $\mathbf{A}\left(\nabla\cdot\mathbf{C}\right)-\mathbf{C}\left(\nabla\cdot\mathbf{A}\right)=\mathbf{C}\left(\int\frac{\rho}{\varepsilon}dt\right)$. Whereas the last two terms, using the curl expressions for the potentials (4) and (5) give

$$\begin{aligned}\left(\mathbf{A}\cdot\nabla^2\mathbf{C}-\mathbf{C}\cdot\nabla^2\mathbf{A}\right)=\\ \nabla\cdot\left[\mathbf{C}\left(\int\frac{\rho}{\varepsilon}dt\right)+\mathbf{A}\times\left(\mu\varepsilon\partial_t\mathbf{A}-\mu\int\mathbf{J}dt\right)-\mathbf{C}\times\left(-\partial_t\mathbf{C}\right)\right].\end{aligned}$$

The term on the left-hand side of (10) involving second temporal derivatives can be rewritten as

$$\mathbf{C}\cdot\partial_t^2\mathbf{A}-\mathbf{A}\cdot\partial_t^2\mathbf{C}=\partial_t\left(\mathbf{C}\cdot\partial_t\mathbf{A}-\mathbf{A}\cdot\partial_t\mathbf{C}\right).$$

Equation (10) then takes the form of a conservation equation

$$\begin{aligned}\nabla\cdot\left(\mu\varepsilon\mathbf{A}\times\partial_t\mathbf{A}+\mathbf{C}\times\partial_t\mathbf{C}\right)+\mu\varepsilon\partial_t\left(\mathbf{C}\cdot\partial_t\mathbf{A}-\mathbf{A}\cdot\partial_t\mathbf{C}\right)=\\ \nabla\cdot\left[-\mathbf{C}\left(\int\frac{\rho}{\varepsilon}dt\right)+\mu\mathbf{A}\times\int\mathbf{J}dt\right]\\ +\frac{1}{\varepsilon}\mathbf{C}\cdot\int\nabla\rho dt+\mu\mathbf{C}\cdot\mathbf{J}+\mu\mathbf{A}\cdot\int\nabla\times\mathbf{J}dt.\end{aligned}$$

Since $\nabla\cdot\left[\mathbf{C}\left(\int\frac{\rho}{\varepsilon}dt\right)\right]=\frac{1}{\varepsilon}\mathbf{C}\cdot\int\nabla\rho dt$, the terms involving the charge gradient cancel out. From the divergence of a cross product identity $\nabla\cdot\left[\mu\mathbf{A}\times\int\mathbf{J}dt\right]=\mu\int\mathbf{J}dt\cdot\left(\nabla\times\mathbf{A}\right)-\mu\mathbf{A}\cdot\left(\nabla\times\int\mathbf{J}dt\right)$, the continuity equation is then

$$\nabla \cdot \left(\mu\varepsilon \mathbf{A} \times \partial_t \mathbf{A} + \mathbf{C} \times \partial_t \mathbf{C}\right) + \mu\varepsilon \partial_t \left(\mathbf{C} \cdot \partial_t \mathbf{A} - \mathbf{A} \cdot \partial_t \mathbf{C}\right) = \mu \int \mathbf{J} dt \cdot (\nabla \times \mathbf{A}) + \mu \mathbf{C} \cdot \mathbf{J}. \tag{11}$$

Therefore there exists a conserved quantity

$$\varrho_{\mathbf{AC}} = \mu\varepsilon \left(\mathbf{C} \cdot \partial_t \mathbf{A} - \mathbf{A} \cdot \partial_t \mathbf{C}\right), \tag{12}$$

with its corresponding flux

$$\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon \mathbf{A} \times \partial_t \mathbf{A} + \mathbf{C} \times \partial_t \mathbf{C}. \tag{13}$$

The flux is represented by the bold italic symbol $\boldsymbol{J}$ and sub-indices indicating the fields involved, whereas electric current is denoted by a bold $\mathbf{J}$ letter. If the time derivatives of the vector potentials are substituted in terms of the fields,

$$\nabla \cdot \left(\mu\varepsilon \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}\right) + \mu\varepsilon \partial_t \left(\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}\right) = \mu \mathbf{B} \cdot \int \mathbf{J} dt + \mu \mathbf{C} \cdot \mathbf{J}. \tag{14}$$

The locally conserved quantity is the *optical helicity density*

$$\varrho_{\mathbf{AC}} = \mu\varepsilon \left(\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}\right). \tag{15}$$

This density is a time-even pseudo scalar that changes sign upon reflection. Regarding spatial symmetry, it is the product of a polar vector, $\mathbf{A}$ or $\mathbf{E}$, times a pseudo vector, $\mathbf{B}$ or $\mathbf{C}$ correspondingly. The temporal symmetry is the product of two time odd vectors $\mathbf{A}, \mathbf{B}$ and two time even vectors $\mathbf{C}, \mathbf{E}$. To accommodate for various interpretations, a factor of $\frac{1}{2}$ is often introduced and the coefficient $\mu\varepsilon$ is either omitted or rescaled in the definitions of different authors. The integral form $\mathcal{H} = \int (\mathbf{A} \cdot \mathbf{B})\, dV$ is a topological invariant commonly used in magnetostatics. This form has been preferred over a differential form because it is gauge invariant if the fields decrease sufficiently fast at infinity [14]. The magnetostatic limit is readily obtained from the above result; If $\mathbf{B} = \nabla \times \mathbf{A}$ is time independent, $\mathbf{A}$ is time independent, thus $\mathbf{C} \cdot \mathbf{E} = -\mathbf{C} \cdot \partial_t \mathbf{A} = 0$ and the magnetostatic helicity density is $\varrho_{\mathbf{AC}} = \mu\varepsilon \mathbf{A} \cdot \mathbf{B}$. The differential form now permits integration over an arbitrary region, including a volume that spans over the whole space where gauge invariance is insured.

The *optical helicity density flux* is

$$\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}. \tag{16}$$

This flux is a time odd pseudo vector, sum of the cross product of two polar vectors plus the cross product of two pseudo vectors. The integral form of (16) with a factor of $\frac{1}{2}$ has been considered to represent the total spin [10], $S = \frac{1}{2} \int (\mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C})\, dV$. This quantity is an intrinsic quantity of the field because it does not depend on external factors such as the choice of a calculation axis. The orbital part of the AM was initially considered an extrinsic property due to the explicit position dependence of the orbital momentum. However, the orbital AM also has an intrinsic part if the transverse momentum is zero [13]. A non paraxial analysis of structured beams reveals a contribution to the 'spin' part weighted by a factor that involves the field's angular spectrum [27]. Furthermore, there is an interconversion of optical spin and orbital AM in strongly focused optical beams [28]. For these reasons, we should reconsider whether the helicity flow represents only the spin part of the AM. It cannot be ruled out that a helical phase front or other features of the field besides its polarization contribute to $\boldsymbol{J}_{\mathbf{AC}}$ in the non paraxial regime.

The early definitions of optical helicity and its flow involved only the leading terms in the above expressions, ie. $\varrho_{\mathbf{AC}} = \mu\varepsilon\mathbf{A}\cdot\mathbf{B}$ and $\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon\mathbf{E}\times\mathbf{A}$. This so called false asymmetric definitions were substituted by the dual-symmetric definitions [15]. In the present derivation, there is no need to invoke a principle of electric-magnetic democracy [29] in order to fulfill this symmetry. The continuity equation reveals that the density and its flow must have the two terms. It is remarkable that research evolved in the right direction in the past ten years introducing the correct terms, albeit with non rigorous arguments, an accomplishment evocative of Koestler's scientific sleepwalkers thesis [30].

It is sometimes convenient to perform a Helmholtz decomposition in longitudinal ($\nabla\times V_{\parallel} = 0$) and transverse ($\nabla\cdot V_{\perp} = 0$) components. Several authors have defined helicity and flux solely in terms of the transverse components of the vector potentials in order to provide gauge independent quantities in the integral form [20]. In the present proposal, the vector potential $\mathbf{C}$ is transverse. The conservation equation replacing $\mathbf{A} = \mathbf{A}_{\perp} + \mathbf{A}_{\parallel}$ in the wave equation (9) and recreating the complementary fields procedure gives

$$\nabla\cdot\left([\mu\varepsilon\mathbf{E}\times\mathbf{A}_{\perp} + \mathbf{B}\times\mathbf{C}_{\perp}] + \mu\varepsilon\mathbf{E}\times\mathbf{A}_{\parallel}\right) + \\ \mu\varepsilon\partial_t\left([\mathbf{A}_{\perp}\cdot\mathbf{B} - \mathbf{C}_{\perp}\cdot\mathbf{E}] + \mathbf{A}_{\parallel}\cdot\mathbf{B}\right) = \mu\mathbf{B}\cdot\int\mathbf{J}dt + \mu\mathbf{C}_{\perp}\cdot\mathbf{J},$$

where the terms in square brackets are the helicity and its flow defined solely in terms of the transverse potentials. Evaluating the divergence and the time derivative of the terms involving $\mathbf{A}_{\parallel}$, the continuity equation can be written as

$$\nabla\cdot[\mu\varepsilon\mathbf{E}\times\mathbf{A}_{\perp} + \mathbf{B}\times\mathbf{C}_{\perp}] + \mu\varepsilon\partial_t[\mathbf{A}_{\perp}\cdot\mathbf{B} - \mathbf{C}_{\perp}\cdot\mathbf{E}] = \\ -\mu\varepsilon\partial_t\mathbf{A}_{\parallel}\cdot\mathbf{B} + \mu\mathbf{B}\cdot\int\mathbf{J}dt + \mu\mathbf{C}_{\perp}\cdot\mathbf{J}. \tag{17}$$

In the present description, there is no way to establish a continuity equation involving only the transverse part of the potentials if $\mathbf{A}_{\parallel}\neq 0$ and its time derivative is not perpendicular to the magnetic field. If the longitudinal component is set equal to zero, $\nabla\cdot\mathbf{A} = \nabla\cdot\mathbf{A}_{\parallel} = 0$, the Coulomb gauge must be enforced and a scalar potential has to be included. Only in the absence of net free charges, the scalar potential can be set to zero. These results are in accordance with recent statements that a definition in terms of (gauge invariant) transverse potentials is consistent with a canonical definition in the Coulomb gauge for free fields [31, 32]. If only the transverse electric field component $\mathbf{E}_{\perp}$, is considered in the definitions of the helicity and its flux in (17), a term $\mu\varepsilon\mathbf{C}\cdot\partial_t\mathbf{E}_{\parallel}$ is also present within the sources. The $-\mu\varepsilon\partial_t\mathbf{A}_{\parallel}\cdot\mathbf{B}$ term in (17) does not have the form of a source but corresponds to the $\mathbf{E}\cdot\mathbf{B}$ term that is zero in the radiation zone (so called orthogonal fields) [33, 34]. This term has also been related to the chiral anomaly of massless fermions [35]. If the helicity is considered to be solely a property of the fully transverse field, it is then only defined far away from the sources where the longitudinal terms are zero [36]. Two further attributes of the transverse fields definitions are that: i) the transverse part is gauge independent only if all the involved fields fall to zero sufficiently fast at spatial infinity [37]. This condition is certainly satisfied in QCD but not necessarily in QED, since plane traveling waves are commonplace in optics. ii) The helicity and its flux are not Lorentz covariant since transverse fields do not transform covariantly [31]. However, if non transverse definitions of the helicity density and its flow are employed, the lack of gauge invariance becomes again an ostensibly unsurmountable problem. This issue will be addressed in a forthcoming communication.

The helicity source terms can be written in the nicely symmetrical form $\mu\mathbf{B}\cdot\int\mathbf{J}dt - \mu\int\mathbf{B}dt\cdot\mathbf{J}$. These terms do not depend on the charge or the charge gradient but only on the current density. This result is reminiscent of Poynting's theorem, where the source of energy density is $\mathbf{E}\cdot\mathbf{J}$, the magnetic field now playing a somewhat analogous role to the electric field. The source terms are zero if the magnetic field and the current are perpendicular. Even if the projection of the magnetic field onto the current direction is not zero, if the magnetic field and the current are linearly dependent, the sum of the source terms is zero. A magnetic field with say, a sine phase dependence and a current with a cosine phase dependence will render non zero source terms.

Let us appraise the helicity and its flow for some simple polarization states. Consider a plane monochromatic wave with frequency ω propagating in the z direction. Linear polarization: Let the fields be

$$\mathbf{E} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{e}}_x, \qquad \mathbf{B} = \frac{n}{c} E_0 \cos(kz - \omega t)\, \hat{\mathbf{e}}_y,$$

where $\mathbf{B} = \frac{n}{c}\hat{\mathbf{e}}_z \times \mathbf{E}$, n is the refractive index and $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are unit Cartesian vectors. The corresponding vector potentials, most easily obtained from time integration of (6) and (7), are

$$\mathbf{A} = \frac{1}{\omega} E_0 \sin(kz - \omega t)\, \hat{\mathbf{e}}_x, \qquad \mathbf{C} = \frac{nE_0}{c\omega} \sin(kz - \omega t)\, \hat{\mathbf{e}}_y.$$

All four terms $\mathbf{A}\cdot\mathbf{B}$, $\mathbf{C}\cdot\mathbf{E}$, $\mathbf{E}\times\mathbf{A}$ and $\mathbf{B}\times\mathbf{C}$ evaluate to zero. $\varrho_{\mathbf{AC}}$ and $\boldsymbol{J}_{\mathbf{AC}}$ are both zero and the conservation equation holds trivially. Elliptical polarization: Consider EM fields that rotate in the $x-y$ plane

$$\mathbf{E} = E_x \cos(kz - \omega t)\, \hat{\mathbf{e}}_x \pm E_y \sin(kz - \omega t)\, \hat{\mathbf{e}}_y, \tag{18}$$

and

$$\mathbf{B} = \mp\frac{nE_y}{c} \sin(kz - \omega t)\, \hat{\mathbf{e}}_x + \frac{nE_x}{c} \cos(kz - \omega t)\, \hat{\mathbf{e}}_y. \tag{19}$$

Circular polarization is of course obtained for $E_x = E_y$. The upper sign corresponds to right hand circularly polarized light with clockwise rotation. The corresponding vector potentials are

$$\mathbf{A} = \frac{1}{\omega} E_x \sin(kz - \omega t)\, \hat{\mathbf{e}}_x \mp \frac{1}{\omega} E_y \cos(kz - \omega t)\, \hat{\mathbf{e}}_y, \tag{20}$$

$$\mathbf{C} = \pm\frac{nE_y}{c\omega} \cos(kz - \omega t)\, \hat{\mathbf{e}}_x + \frac{nE_x}{c\omega} \sin(kz - \omega t)\, \hat{\mathbf{e}}_y. \tag{21}$$

The inner products are $\mathbf{A}\cdot\mathbf{B} = \mp\frac{n}{\omega c}E_xE_y$, $\mathbf{C}\cdot\mathbf{E} = \pm\frac{n}{\omega c}E_xE_y$, both terms contribute with equal weight,

$$\varrho_{\mathbf{AC}} = \frac{n^2}{c^2}\left(\mathbf{A}\cdot\mathbf{B} - \mathbf{C}\cdot\mathbf{E}\right) = \mp\frac{2n^3}{\omega c^3} E_x E_y. \tag{22}$$

Right hand circularly polarized light has negative helicity (upper sign) in accordance with the terminology of modern physics [5, 38, p.176]. Whereas the cross products are $\mathbf{E}\times\mathbf{A} = \mp\frac{1}{\omega}E_xE_y\hat{\mathbf{e}}_z$, $\mathbf{B}\times\mathbf{C} = \mp\frac{n^2E_xE_y}{c^2\omega}\hat{\mathbf{e}}_z$. The flow is

$$\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C} = \mp\frac{2n^2}{c^2\omega}E_xE_y\hat{\mathbf{e}}_z. \tag{23}$$

In terms of the eccentricity,

$$\varrho_{\mathbf{AC}} = \mp\frac{2n^3}{c^3\omega}E_0^2\sqrt{1-e^2},\ \boldsymbol{J}_{\mathbf{AC}} = \mp\frac{2n^2}{c^2\omega}E_0^2\sqrt{1-e^2}\hat{\mathbf{e}}_z, \tag{24}$$

maximum helicity flow is obtained for circular polarization ($e = 0$), it falls to zero as the ellipse degenerates onto a straight line segment ($e = 1$). Reassuringly, the flow is equal to the helicity density times the velocity of propagation, $\boldsymbol{J}_{\mathbf{AC}} = \varrho_{\mathbf{AC}} \left(\frac{c}{n}\hat{\mathbf{e}}_z\right)$. Notice that these plane wave solutions do not admit sources. The current $\mu\mathbf{J} = \nabla \times \mathbf{B} - \mu\varepsilon\partial_t\mathbf{E}$, obtained from the field functions (18), (19) is zero. In general, the direction of the helicity flow $\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C}$ need not be parallel to $\mathbf{E}\times\mathbf{H}$. Thus, the helicity flux density can also have a transverse component with respect to the wave propagation. These sometimes called photonic wheels, have indeed been predicted and observed in evanescent as well as propagating waves of some structured fields [19, 39].

The present derivation has been deliberately performed using only real expressions for the fields. It has been pointed out that some conservation equations with complex time dependent fields [40] hold only for monochromatic fields with time-independent complex amplitudes [31]. The representation of fields with complex algebra although straight forward in the linear domain, requires careful assessment if field products are involved. The present treatment clearly holds for optical pulses or polychromatic fields in dispersionless homogeneous media.

The expressions for $\varrho_{\mathbf{AC}}$ and $\boldsymbol{J}_{\mathbf{AC}}$ involve no cycle averaging, nonetheless, the results are space and time independent for plane waves with arbitrary elliptical or plane polarization. This feature has been noted before but no physical explanation has been given [24, 36]. The complementary fields provide an adequate physical explanation: The content in one field is transferred to the other field such that the total field content is constant. There is a dynamical equilibrium whereby the content moves to and fro the two fields. For example, consider the $\mathbf{A}\cdot\mathbf{B} = -\mathbf{A}\cdot\partial_t\mathbf{C}$ term, the $\mathbf{A}$ field is out of phase with respect to the $\mathbf{C}$ field, as may be seen from (20) and (21). The $\mathbf{A}$ field $\hat{\mathbf{e}}_x$ component increases while the $\mathbf{C}$ field $\hat{\mathbf{e}}_x$ component decreases and vice versa. A similar behaviour is displayed by the $\hat{\mathbf{e}}_y$ components of these two fields. The field content is thus exchanged between the two of them, the product $\mathbf{A}\cdot\partial_t\mathbf{C}$ being a measure of this exchange. An analogous dynamical equilibrium is exhibited by the $\mathbf{C}\cdot\mathbf{E} = -\mathbf{C}\cdot\partial_t\mathbf{A}$ term.

Forget for a moment about the separation of the AM in spin and orbital parts. Consider that all we have is the above complementary fields electromagnetic derivation of $\varrho_{\mathbf{AC}}$ and $\boldsymbol{J}_{\mathbf{AC}}$ (where the mechanical definition $\mathbf{r}\times\mathbf{p}$ was nowhere invoked). Do we have enough elements to consider that $\boldsymbol{J}_{\mathbf{AC}}$ represents an angular momentum? On the one hand, we have the spatial and temporal symmetries: $\boldsymbol{J}_{\mathbf{AC}}$ is a time odd pseudo vector, a behavior consistent with a quantity representing angular momentum. If a wave is plane polarized, the helicity $\varrho_{\mathbf{AC}}$ and its flow $\boldsymbol{J}_{\mathbf{AC}}$ are zero. They become different from zero for elliptical polarization and have opposite signs for right and left circular polarization. So, it is reasonable to consider that the helicity and its flow are indeed a measure of the intrinsic rotational content of the field. However, whether the helicity flow represents angular momentum is not conclusive. It has been argued that $\boldsymbol{J}_{\mathbf{AC}}$ has units of AM. However, since the continuity equation can be multiplied by different constants, dimensional analysis is not unequivocal in this case. The ω factors in $\varrho_{\mathbf{AC}}$ and $\boldsymbol{J}_{\mathbf{AC}}$ have been discussed from the outset of the problem [41]. Different results are obtained depending on which of the field amplitudes is considered frequency independent. If the electric field amplitude is ω independent, the results reported here are obtained. However, if the starting point is $\mathbf{A}$ with ω independent amplitude as in [42, 43], then $A_x = \frac{1}{\omega}E_x$, $A_y = \frac{1}{\omega}E_y$ and from (23), $\boldsymbol{J}_{\mathbf{AC}} \propto \omega A_x A_y$. In QED, the electric field amplitude is proportional to $\sqrt{\omega}$, then $E_x = \sqrt{\omega}A_x$, $E_y = \sqrt{\omega}A_y$ and from (23), $\boldsymbol{J}_{\mathbf{AC}}$ is ω independent, consistent with the quantum treatment where the photon AM is equal to $\pm\hbar$.

Acknowledgment: The author thanks the careful revision of the manuscript by J. Hernández.

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