Chirality, helicity and the rotational content of electromagnetic fields

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Abstract

The chirality continuity equation in the presence of charge and current is obtained from the complementary fields formalism. In the presence of sources, it is strongly suggested that the chirality should be defined proportional to $\mu\varepsilon\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)$ instead of the usual definition $\mathbf{H}\cdot(\nabla\times\mathbf{B})+\varepsilon\mathbf{E}\cdot(\nabla\times\mathbf{E})$. Both definitions are equivalent for free fields. The relationship between helicity $\varrho_{\mathbf{AC}}$ and chirality $\varrho_{\mathbf{EH}}$ for arbitrary wave solutions is shown to be $\partial_t^2\varrho_{\mathbf{AC}} = \mu\varepsilon\left(\frac{1}{\varepsilon}\varrho_{\mathbf{EH}}+\mathbf{A}\cdot\partial_t^2\mathbf{B}-\mathbf{C}\cdot\partial_t^2\mathbf{E}\right)$. The helicity continuity equation can be transformed into the chirality scheme via a time derivative transformation of all the field functions. Following a mechanical analogue, the helicity flow is shown to be equivalent to the angular momentum $\boldsymbol{J}_{\mathbf{rp}}=\mathbf{r}\times\mathbf{p}$, while the chirality flow is equivalent to $\boldsymbol{J}_{\mathbf{vF}}\equiv\dot{\mathbf{r}}\times\dot{\mathbf{p}}$. Both quantities are shown to adequately describe the rotational content of the electromagnetic field.

Keywords: Electromagnetic angular momentum, Chirality, Helicity, Conservation equations.

Declarations of interest: none

1. Introduction

The intrinsic rotational content of an electromagnetic field has been described via the optical helicity or the optical chirality. Which of the two quantities adequately describes the angular momentum (AM) content of the field is subject of intense debate. The starting point to the helicity expression is the mechanical definition of angular momentum translated to the electromagnetic domain $\boldsymbol{J}=\mathbf{r}\times\mathbf{p}=\mathbf{r}\times\mathbf{S}$, where $\mathbf{r}$ represents position and the linear momentum $\mathbf{p}$ is equal to Poynting's vector $\mathbf{S}$. In order to separate the spin and orbital parts, a procedure not exempt of controversy is used [1, 2]. The magnetic field is written in terms of the potential

$\mathbf{E}\times(\nabla\times\mathbf{A})$ and the AM is written as $\boldsymbol{J}=[\mathbf{E}\cdot(\mathbf{r}\times\nabla)\,\mathbf{A}]-(\mathbf{E}\cdot\nabla)\,(\mathbf{r}\times\mathbf{A})+\mathbf{E}\times\mathbf{A}$. The first term is associated with orbital angular momentum and the last with spin or polarization angular momentum. Spatial integration of this equation over all space is needed to dismiss the mid term. For this reason, the integral form of the spin or intrinsic angular momentum $\int\mathbf{E}\times\mathbf{A}dr^3$ is preferred. An advantage of this procedure is that the $\mathbf{E}\times\mathbf{A}$ term is no longer dependent on the position vector $\mathbf{r}$, and can thus be related to the intrinsic angular momentum of an EM wave. The lack of electric-magnetic rotation symmetry has been remedied by patching this expression to $\int(\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})\,dr^3$ [3], invoking a 'democracy' principle [4, 5]. A different formulation based on currents conservation, was followed much earlier by Afanasiev and Stepanovsky to obtain an identical result [6]. The main concern with this approach has been gauge invariance. To insure invariance, the usual procedure has been to define the spin in terms of the transverse potentials and fields [7, 8, 9]. An alternative gauge invariant continuity equation obtained by the complementary fields formalism has been recently proposed [10]. The corresponding helicity or projection of the spin onto the propagation direction is $\int(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E})\,dr^3$.

From the electromagnetic field tensor $F^{\alpha\beta}$ and its Hodge dual ${}^{\star}F^{\mu\lambda}$, the relationship $Z^{\mu}_{\nu\rho}={}^{\star}F^{\mu\lambda}\partial_\rho F_{\lambda\nu}-F^{\mu\lambda}\partial_\rho{}^{\star}F_{\lambda\nu}$ establishes a set of EM conserved quantities [11, 12, 13]. This procedure can be extended to obtain an infinite set of conservation equations [14, 15]. One of these conserved quantities is the time-even pseudo-scalar, $\frac{1}{2\mu}\mathbf{B}\cdot(\nabla\times\mathbf{B})+\frac{\varepsilon}{2}\mathbf{E}\cdot(\nabla\times\mathbf{E})$. This density has the required parity for interactions with chiral molecules and was given the physical significance of optical chirality [16].The corresponding chirality flow is $\frac{1}{2\mu}\mathbf{E}\times(\nabla\times\mathbf{B})-\frac{1}{2\mu}\mathbf{B}\times(\nabla\times\mathbf{E})$. A quantum version of the optical chirality has been shown to be a measure of the spin angular momentum [17]. Conservation of this quantity has also been obtained from a simple symmetry of the electromagnetic action using Noether's theorem. The symmetry transformation of the vector potential is $\delta A=\eta\nabla\times\partial_t A$, $\delta\phi=0$, where η is the infinitesimal parameter of the transformation [18]. The chiral content of various types of optical pulses has been recently undertaken [19]. The emphasis of the chiral approach, not surprisingly since it is defined in terms of observables, has been in the EM wave interaction with matter, in particular interaction with enantiomers [20]. However, there are two important drawbacks to this approach. On the one hand, it is not derived from an angular momentum related expression as a starting point. On the other, there is an unexpected frequency dependence of the involved quantities. This issue, has also been related to the units of the chirality and its flow.

In this communication, the chirality conservation equation with sources is derived in section 2, using the complementary fields formalism. A novel proposal for the definition of chirality is presented and is compared with previous definitions. Plane wave elliptical polarization states are exemplified in section 3. It is shown that the assessed quantities are space and time independent although no averages are performed. This result is physically explained via the energy exchange of the complementary fields involved. The general relationship between chirality and helicity is discussed in section 4. In the first part of this section, the previously known proportionality for harmonic waves is extended to arbitrary waves. In the second part, a deeper analysis is undertaken establishing the relationship between the chirality and helicity partial differential equation sets. Final remarks, including a mechanical analogue of both, chirality and helicity, are drawn in the last section. An explanation why expressions of the form $\mathbf{E} \times \partial_t \mathbf{E}$ and $\mathbf{E} \cdot \partial_t \mathbf{H}$ are related to the rotational content of the field is also expounded in this section.

2. Chirality continuity equations

A continuity equation may be obtained from two complementary fields, that is, two linearly independent vector fields. From a physical point of view, the fields are complementary because the energy content of the system is dynamically exchanged between these two fields. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are the two fields involved. A one dimensional version of this procedure with scalar fields has been successfully implemented for scalar wave phenomena [21]. This approach has also been implemented in classical mechanics for time dependent linear restoring force systems such as the harmonic oscillator [22]. Commence with the classical electromagnetic equations,

$$\nabla \cdot \mathbf{D} = \rho \tag{1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1b}$$

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \tag{1c}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{1d}$$

The wave equation for the electric field in an electric and magnetic homogeneous linear medium $\mathbf{D} = \varepsilon \mathbf{E}$, with possibly inhomogeneous charge distribution is

$$\nabla^2 \mathbf{E} - \mu\varepsilon \partial_t^2 \mathbf{E} = \frac{1}{\varepsilon} \nabla \rho + \mu \partial_t \mathbf{J}. \tag{2}$$

Whereas the wave equation for the magnetic field with $\mathbf{B} = \mu \mathbf{H}$, is

$$\nabla^2 \mathbf{H} - \mu\varepsilon \partial_t^2 \mathbf{H} = -\nabla \times \mathbf{J}. \tag{3}$$

Evaluate the inner product of the electric field $\mathbf{E}$ with the magnetic field wave equation (3), $\mathbf{E} \cdot \nabla^2 \mathbf{H} - \mu\varepsilon \mathbf{E} \cdot \partial_t^2 \mathbf{H} = -\mathbf{E} \cdot \nabla \times \mathbf{J}$ and the inner product of the magnetic field $\mathbf{H}$ with the wave equation (2) for the electric field, $\mathbf{H} \cdot \nabla^2 \mathbf{E} - \mu\varepsilon \mathbf{H} \cdot \partial_t^2 \mathbf{E} = \frac{1}{\varepsilon} \mathbf{H} \cdot \nabla \rho + \mu \mathbf{H} \cdot \partial_t \mathbf{J}$. The difference of these two equations is

$$\begin{aligned}\left(\mathbf{E} \cdot \nabla^2 \mathbf{H} - \mathbf{H} \cdot \nabla^2 \mathbf{E}\right) &+ \mu\varepsilon \left(\mathbf{H} \cdot \partial_t^2 \mathbf{E} - \mathbf{E} \cdot \partial_t^2 \mathbf{H}\right) \\ &= -\frac{1}{\varepsilon} \mathbf{H} \cdot \nabla \rho - \mu \mathbf{H} \cdot \partial_t \mathbf{J} - \mathbf{E} \cdot \nabla \times \mathbf{J}.\end{aligned} \tag{4}$$

The terms involving the Laplacians can be written, invoking a vector version of Green's second identity [23], as

$$\left(\mathbf{E}\cdot\nabla^2\mathbf{H}-\mathbf{H}\cdot\nabla^2\mathbf{E}\right)=\nabla\cdot\left(\mathbf{E}\left(\nabla\cdot\mathbf{H}\right)-\mathbf{H}\left(\nabla\cdot\mathbf{E}\right)+\mathbf{E}\times\nabla\times\mathbf{H}-\mathbf{H}\times\nabla\times\mathbf{E}\right)$$

and the divergence of the fields can be written in terms of the sources from Maxwell's equations $\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon}$, $\nabla\cdot\mathbf{H}=0$,

$$\left(\mathbf{E}\cdot\nabla^2\mathbf{H}-\mathbf{H}\cdot\nabla^2\mathbf{E}\right)=\nabla\cdot\left(-\frac{\rho}{\varepsilon}\mathbf{H}+\mathbf{E}\times\nabla\times\mathbf{H}-\mathbf{H}\times\nabla\times\mathbf{E}\right).$$

The second order temporal derivatives can be grouped as

$$\mathbf{H}\cdot\partial_t^2\mathbf{E}-\mathbf{E}\cdot\partial_t^2\mathbf{H}=\partial_t\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right).$$

A continuity equation is thus obtained

$$\begin{aligned}\nabla\cdot\left(\mathbf{E}\times\nabla\times\mathbf{H}-\mathbf{H}\times\nabla\times\mathbf{E}\right)+\mu\varepsilon\partial_t\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)&\\ =-\mu\mathbf{H}\cdot\partial_t\mathbf{J}-\mathbf{E}\cdot\nabla\times\mathbf{J}.&\end{aligned}\tag{5}$$

where $\nabla\cdot\left(\frac{\rho}{\varepsilon}\mathbf{H}\right)=\frac{\rho}{\varepsilon}\nabla\cdot\mathbf{H}+\nabla\frac{\rho}{\varepsilon}\cdot\mathbf{H}$, but $\nabla\cdot\mathbf{H}$ is zero and the second term is canceled out with its counterpart on the RHS of (4). From here onwards two different procedures will be pursued, one involves writing the fields in terms of time derivatives whereas the other states the problem in terms of spatial derivatives.

2.1. Fields' time derivatives

From (1c) and (1d), write the curls within the divergence in the continuity equation (5), in terms of the field's time derivatives,

$$\begin{aligned}\nabla\cdot\left(\mathbf{E}\times\left(\partial_t\mathbf{D}+\mathbf{J}\right)+\mathbf{H}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)&\\ =-\mu\mathbf{H}\cdot\partial_t\mathbf{J}-\mathbf{E}\cdot\nabla\times\mathbf{J}.&\end{aligned}$$

Grouping all terms involving the source current $\mathbf{J}$ on the RHS and recalling that $\nabla\cdot\left(\mathbf{E}\times\mathbf{J}\right)=\mathbf{J}\cdot\left(\nabla\times\mathbf{E}\right)-\mathbf{E}\cdot\left(\nabla\times\mathbf{J}\right)$, the conservation equation becomes

$$\begin{aligned}\nabla\cdot\left(\mathbf{E}\times\partial_t\mathbf{D}+\mathbf{H}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)&\\ =\mu\left(\mathbf{J}\cdot\partial_t\mathbf{H}-\mathbf{H}\cdot\partial_t\mathbf{J}\right).&\end{aligned}\tag{6a}$$

The conserved density is then

$$\varrho_{\mathbf{EH}}=\mu\varepsilon\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right),\tag{6b}$$

and its flux is

$$\boldsymbol{J}_{\mathbf{EH}}=\mathbf{E}\times\partial_t\mathbf{D}+\mathbf{H}\times\partial_t\mathbf{B}.\tag{6c}$$

The continuity equation is sometimes written solely in terms of $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{aligned}\nabla\cdot\left(\mu\varepsilon\mathbf{E}\times\partial_t\mathbf{E}+\mathbf{B}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\mathbf{B}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{B}\right)&\\ =\mu\left(\mathbf{J}\cdot\partial_t\mathbf{B}-\mathbf{B}\cdot\partial_t\mathbf{J}\right),&\end{aligned}\tag{6d}$$

with the concomitant rescaled definitions of density $\varrho_{\mathbf{EB}}$ and flux $\boldsymbol{J}_{\mathbf{EB}}$.

2.2. Fields' space derivatives

In this alternative, write the time derivatives of the fields in Eq. (5), in terms of curls of the fields,

$$\nabla\cdot(\mathbf{E}\times\nabla\times\mathbf{H}-\mathbf{H}\times\nabla\times\mathbf{E})+\mu\varepsilon\partial_t\left(\frac{1}{\varepsilon}\mathbf{H}\cdot(\nabla\times\mathbf{H}-\mathbf{J})+\frac{1}{\mu}\mathbf{E}\cdot(\nabla\times\mathbf{E})\right)$$
$$=-\mu\mathbf{H}\cdot\partial_t\mathbf{J}-\mathbf{E}\cdot\nabla\times\mathbf{J}.$$

Grouping terms involving $\mathbf{J}$ on the sources side of the equation (RHS),

$$\nabla\cdot(\mathbf{E}\times\nabla\times\mathbf{H}-\mathbf{H}\times\nabla\times\mathbf{E})+\partial_t\left(\varepsilon\mathbf{E}\cdot(\nabla\times\mathbf{E})+\mu\mathbf{H}\cdot\nabla\times\mathbf{H}\right)$$
$$=-\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{J}-\mathbf{E}\cdot\nabla\times\mathbf{J} \quad (7a)$$

In terms of the fields $\mathbf{E}$ and $\mathbf{B}$, the chirality density is then

$$\varrho_{\mathbf{EH}}^{\nabla\times}\equiv\frac{1}{\mu}\mathbf{B}\cdot(\nabla\times\mathbf{B})+\varepsilon\mathbf{E}\cdot(\nabla\times\mathbf{E})\,, \quad (7b)$$

and its flow

$$\boldsymbol{J}_{\mathbf{EH}}^{\nabla\times}\equiv\frac{1}{\mu}\mathbf{E}\times(\nabla\times\mathbf{B})-\frac{1}{\mu}\mathbf{B}\times(\nabla\times\mathbf{E})\,. \quad (7c)$$

These are the expressions used by Tang and Cohen (save for a factor of $\frac{1}{2}$) [16]. If the $\mathbf{H}$ and $\mathbf{D}$ fields are retained, the chirality density is $\varrho_{\mathbf{EH}}^{\nabla\times}=\mathbf{H}\cdot(\nabla\times\mathbf{B})+\mathbf{E}\cdot(\nabla\times\mathbf{D})$, expressions of this form have been recently proposed to describe propagation in dispersive media with absorption [24]. An helicity operator has also been proposed to cope with dispersive inhomogeneous media [25].

2.3. Comparison

In order to compare the chirality density definitions, substitute the time derivatives in $\varrho_{\mathbf{EH}}$ given by (6b) in terms of the curls

$$\varrho_{\mathbf{EH}}=\mu\varepsilon\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)=\mu\varepsilon\left(\mathbf{H}\cdot\frac{1}{\varepsilon}\left(\nabla\times\mathbf{H}-\mathbf{J}\right)-\mathbf{E}\cdot\left(-\frac{1}{\mu}\nabla\times\mathbf{E}\right)\right),$$

and rewrite the result in terms of $\varrho_{\mathbf{EH}}^{\nabla\times}$. Follow an analogous procedure for the flows. The relationship between the two proposals is then

$$\varrho_{\mathbf{EH}}=\varrho_{\mathbf{EH}}^{\nabla\times}-\mathbf{B}\cdot\mathbf{J}. \quad (8a)$$

and

$$\boldsymbol{J}_{\mathbf{EH}}=\boldsymbol{J}_{\mathbf{EH}}^{\nabla\times}-\mathbf{E}\times\mathbf{J}. \quad (8b)$$

In the absence of current the two definitions are identical. For this reason, on occasions the two definitions are used indistinctly [26, 19]. Regarding the sources, notice that the first term is equal in either case since $\mu\mathbf{J}\cdot\partial_t\mathbf{H}=-\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{J}$.

However, the other source term is clearly different, from the difference of the continuity equations (6a) and (7a),

$$\nabla \cdot \left(\boldsymbol{J}_{\mathbf{EH}} - \boldsymbol{J}_{\mathbf{EH}}^{\nabla\times} \right) + \partial_t \left(\varrho_{\mathbf{EH}} - \varrho_{\mathbf{EH}}^{\nabla\times} \right) = -\mathbf{B} \cdot \partial_t \mathbf{J} + \mathbf{E} \cdot \nabla \times \mathbf{J}.$$

Substitution of the relationship between densities and fluxes becomes an identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{J}) + \partial_t (\mathbf{B} \cdot \mathbf{J}) = \mathbf{B} \cdot \partial_t \mathbf{J} - \mathbf{E} \cdot \nabla \times \mathbf{J}.$$

The definition of the helicity and consequently its flux have followed the expressions of fluid dynamics. The hydrodynamic helicity is defined as $\mathcal{H} = \frac{1}{2} \int_V \mathbf{u} \cdot \boldsymbol{\omega} \, d^3 r$, where $\mathbf{u}$ represents the streamlines (trajectories of the velocities at constant time); the vorticity is given by the curl of the streamlines $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ [27, and refs. therein]. If the flow is steady, the particle paths coincide with the streamlines [28]. In the magnetostatic limit, an analogous expression regarding topological properties of the magnetic helicity $\int (\mathbf{A} \cdot \mathbf{H}) \, d^3 r$, with $\mathbf{H} = \mu^{-1} \nabla \times \mathbf{A}$, was also adopted in magentohydrodynamics [29, 6]. In both cases, $\mathbf{u}$ or $\mathbf{A}$ are usually either independent or slowly varying functions of time. It is then appropriate to describe the problem in terms of these variables. However, this is not the case in electrodynamics where the spatial and temporal derivatives are related by Ampère and Faraday's expressions in Maxwell's equations and neither can be neglected.

3. Evaluation of simple polarization states

The evaluation of the various polarization states is deliberately performed using only real expressions for the fields. It has been pointed out that some conservation equations with complex time dependent fields [30] hold only for monochromatic fields with time-independent complex amplitudes [8]. The representation of fields with complex algebra although straight forward in the linear domain, requires careful assessment if field products are involved. The present treatment clearly holds for optical pulses or polychromatic fields in dispersionless homogeneous media.

Consider a linearly polarized plane wave propagating in the $\hat{\mathbf{e}}_z$ direction with electric and magnetic fields given by

$$\mathbf{E} = E_0 \cos(kz - \omega t) \, \hat{\mathbf{e}}_x, \quad \mathbf{H} = \sqrt{\frac{\varepsilon}{\mu}} E_0 \hat{\mathbf{k}} \times \mathbf{E} = H_0 \cos(kz - \omega t) \, \hat{\mathbf{e}}_y,$$

where $H_0 = \sqrt{\frac{\varepsilon}{\mu}} E_0$. The dot products $\mathbf{H} \cdot \partial_t \mathbf{E}$ and $\mathbf{E} \cdot \partial_t \mathbf{H}$ are both zero since the involved vectors are perpendicular and thus, the chirality is zero $\varrho_{\mathbf{EH}} = 0$. For the flow, $\mathbf{E} \times \partial_t \mathbf{D}$ and $\mathbf{H} \times \partial_t \mathbf{B}$ are both zero because the vectors in both factors are parallel, the chirality flow is also zero $\boldsymbol{J}_{\mathbf{EH}} = 0$, as expected. Consider an elliptically polarized wave with the electric field represented by

$$\mathbf{E} = E_{0x} \cos(kz - \omega t) \, \hat{\mathbf{e}}_x \pm E_{0y} \sin(kz - \omega t) \, \hat{\mathbf{e}}_y, \tag{9}$$

where $E_0 = \sqrt{E_{0x}^2 + E_{0y}^2}$. The corresponding magnetic field is

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{1}{\mu c}\hat{\mathbf{k}} \times \mathbf{E} = \mp H_{0x} \sin\left(kz - \omega t\right) \hat{\mathbf{e}}_x + H_{0y} \cos\left(kz - \omega t\right) \hat{\mathbf{e}}_y, \qquad (10)$$

where $H_{0x} = \sqrt{\frac{\varepsilon}{\mu}} E_{0y}$ and $H_{0y} = \sqrt{\frac{\varepsilon}{\mu}} E_{0x}$, the proportionality coefficient is the inverse of the vacuum impedance Z_0. The so called electric contribution (in analogy to the $\mathbf{C} \cdot \mathbf{E}$ electric helicity) to the chirality term is

$$\mathbf{H} \cdot \partial_t \mathbf{E} = \mp E_{0x} H_{0x} \omega \sin^2\left(kz - \omega t\right) \mp E_{0y} H_{0y} \omega \cos^2\left(kz - \omega t\right),$$

whereas the magnetic contribution to the chirality term is (in analogy to the $\mathbf{A} \cdot \mathbf{B}$ magnetic helicity)

$$\mathbf{E} \cdot \partial_t \mathbf{H} = \pm E_{0x} H_{0x} \omega \cos^2\left(kz - \omega t\right) \pm E_{0y} H_{0y} \omega \sin^2\left(kz - \omega t\right).$$

If the magnetic field amplitudes H_{0x}, H_{0y} are written in terms of the electric field amplitudes, each of the above two expressions are time and space independent. However, even if more general relationships were satisfied where $E_{0x} H_{0x} \neq E_{0y} H_{0y}$, for example in anisotropic media, the chirality that involves the difference between both terms, is still space time independent,

$$\varrho_{\mathbf{EH}} = \mu\varepsilon\left(\mathbf{H} \cdot \partial_t \mathbf{E} - \mathbf{E} \cdot \partial_t \mathbf{H}\right) = \mp\mu\varepsilon\left(E_{0x} H_{0x} + E_{0y} H_{0y}\right)\omega = \mp\frac{2\varepsilon}{c} E_{0x} E_{0y} \omega.$$

The two flow terms are of the form

$$\mathbf{E} \times \partial_t \mathbf{D} = \mp\varepsilon E_{0x} E_{0y} \omega \cos^2\left(kz - \omega t\right) \hat{\mathbf{e}}_z \mp \varepsilon E_{0x} E_{0y} \omega \sin^2\left(kz - \omega t\right) \hat{\mathbf{e}}_z = \mp\varepsilon E_{0x} E_{0y} \omega \hat{\mathbf{e}}_z,$$

the $\mathbf{H} \times \partial_t \mathbf{B}$ has a $\mp\mu H_{0x} H_{0y}$ coefficient. The chirality flow is then

$$\boldsymbol{J}_{\mathbf{EH}} = \mathbf{E} \times \partial_t \mathbf{D} + \mathbf{H} \times \partial_t \mathbf{B} = \mp 2\varepsilon E_{0x} E_{0y} \omega \hat{\mathbf{e}}_z.$$

These results can be written in terms of the eccentricity $E_{0x} E_{0y} = E_0^2 \sqrt{1 - e^2}$. Right hand circularly polarized light has negative helicity (upper sign) or negative chirality in this case, in accordance with the usual modern physics convention [31, 32, p.176]. Other polarization states with slightly more complex wavefront structures such as Hermite-Gaussian modes have been recently addressed with this formalism [33].

The expressions for $\varrho_{\mathbf{EH}}$ and $\boldsymbol{J}_{\mathbf{EH}}$ involve no cycle averaging, nonetheless, their values as we have just shown, are space and time independent for plane waves with arbitrary elliptical polarization. Unlike Poynting's theorem for linearly polarized fields, the chirality continuity equation does not require any averaging over time or space. The complementary fields approach can be understood, from a physical point of view, as the dynamical equilibrium between two fields. In this case, the $\mathbf{E}$ and $\mathbf{H}$ elliptically polarized fields exchange their energy in the $\hat{\mathbf{e}}_x$ as well as the $\hat{\mathbf{e}}_y$ directions, as may be seen from (9) and (10). These fields are $\pi/2$

out of phase in each direction. The conserved density, in this case, the chirality, is a measure of the energy exchange between the two fields. Notice that when the two fields are linearly dependent, i.e. linearly polarized, the chirality is zero. This feature is also present in the helicity and spin content of EM fields [3, 34]. The complementary fields are then the $\mathbf{A}$ and $\mathbf{C}$ out of phase potential fields [10]. The locally conserved quantities arising from the complementary fields prevail without any averaging over time or space. This attribute is a hallmark of the complementary fields structure, whereby the energy content of the fields is dynamically transposed between them.

4. Relationship between chirality and helicity

4.1. Helicity continuity equation

The helicity continuity equation including electric sources is [34]

$$\nabla\cdot(\mu\varepsilon\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C})+\mu\varepsilon\partial_t\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right)=\mu\mathbf{B}\cdot\int\mathbf{J}dt-\mu\int\mathbf{B}dt\cdot\mathbf{J}. \quad (11a)$$

The locally conserved quantity is the *optical helicity density*

$$\varrho_{\mathbf{AC}}=\mu\varepsilon\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right). \quad (11b)$$

and the *optical helicity density flux* is

$$\boldsymbol{J}_{\mathbf{AC}}=\mu\varepsilon\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C}. \quad (11c)$$

These quantities are often defined in the literature with a $\frac{1}{2}$ factor and also considering only the transverse fields, natural units are also frequently used [35, 36, and refs. therein]. The electric current source terms can be assigned in different ways in the $\mathbf{A}$ and $\mathbf{C}$ vector potential differential equations [37, 38, 39]. An asset of the assignment leading to (11a) is that it reproduces the structure of Maxwell's equations for the potentials with the corresponding time integrated sources [34].

4.2. Harmonic waves

From the chirality definition (6b), $\varrho_{\mathbf{EH}}=\mu\varepsilon\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right)$, replace $\mathbf{B}=\mu\mathbf{H}$ and in the field time derivatives substitute by the time derivatives of the potentials $\mathbf{E}=-\partial_t\mathbf{A}$ $\mathbf{H}=-\frac{1}{\mu}\partial_t\mathbf{C}$,

$$\varrho_{\mathbf{EH}}=\mu\varepsilon\left(-\frac{\mathbf{B}}{\mu}\cdot\partial_t^2\mathbf{A}+\frac{1}{\mu}\mathbf{E}\cdot\partial_t^2\mathbf{C}\right).$$

If the fields are harmonic, $\partial_t^2\mathbf{A}=-\omega^2\mathbf{A}$ and $\partial_t^2\mathbf{C}=-\omega^2\mathbf{C}$, then $\varrho_{\mathbf{EH}}=\mu\varepsilon\frac{\omega^2}{\mu}\left(\mathbf{B}\cdot\mathbf{A}-\mathbf{E}\cdot\mathbf{C}\right)$, and from the helicity definition,

$$\varrho_{\mathbf{EH}}=\frac{1}{\mu}\omega^2\varrho_{\mathbf{AC}}. \quad (12)$$

The ω^2 factor between chirality and helicity for harmonic waves, has been previously noticed [35, 18].

4.3. Relationship for arbitrary waves

However, let us proceed differently to obtain a general result for waves with any time dependence. Evaluate the temporal derivative of the helicity $\varrho_{\mathbf{AC}}$,

$$\partial_t \varrho_{\mathbf{AC}} = \mu\varepsilon \left(\partial_t \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \partial_t \mathbf{B} - \partial_t \mathbf{C} \cdot \mathbf{E} - \mathbf{C} \cdot \partial_t \mathbf{E} \right).$$

Substitute the derivative of the potentials in term of the fields, The $\mathbf{E}\cdot\mathbf{B}$ first and third terms cancel out, $\partial_t \varrho_{\mathbf{AC}} = \mu\varepsilon \left(\mathbf{A} \cdot \partial_t \mathbf{B} - \mathbf{C} \cdot \partial_t \mathbf{E} \right)$. Evaluate the second time derivative

$$\partial_t^2 \varrho_{\mathbf{AC}} = \mu\varepsilon \left(-\mu \mathbf{E} \cdot \partial_t \mathbf{H} + \mathbf{A} \cdot \partial_t^2 \mathbf{B} + \mu \mathbf{H} \cdot \partial_t \mathbf{E} - \mathbf{C} \cdot \partial_t^2 \mathbf{E} \right),$$

but since the chirality (6b) is $\varrho_{\mathbf{EH}} = \mu\varepsilon \left(\mathbf{H} \cdot \partial_t \mathbf{E} - \mathbf{E} \cdot \partial_t \mathbf{H} \right)$, the relationship between helicity and chirality for arbitrary fields is obtained

$$\partial_t^2 \varrho_{\mathbf{AC}} = \mu\varepsilon \left(\frac{1}{\varepsilon} \varrho_{\mathbf{EH}} + \mathbf{A} \cdot \partial_t^2 \mathbf{B} - \mathbf{C} \cdot \partial_t^2 \mathbf{E} \right). \tag{13}$$

The solution for harmonic waves can be recovered by noticing that $\varrho_{\mathbf{AC}}$ is constant in this case, thus $\partial_t^2 \varrho_{\mathbf{AC}} = 0$, and the above equation becomes $\varrho_{\mathbf{EH}} = -\varepsilon \left(\mathbf{A} \cdot \partial_t^2 \mathbf{B} - \mathbf{C} \cdot \partial_t^2 \mathbf{E} \right)$. For harmonic waves $\partial_t^2 \mathbf{B} = -\omega^2 \mathbf{B}$ and $\partial_t^2 \mathbf{E} = -\omega^2 \mathbf{E}$, so that $\varrho_{\mathbf{EH}} = \varepsilon\omega^2 \left(\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E} \right)$, then (12) is obtained.

Through a similar procedure, the time derivative of the helicity flow $\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}$ is $\partial_t \boldsymbol{J}_{\mathbf{AC}} = \mu \partial_t \mathbf{D} \times \mathbf{A} + \partial_t \mathbf{B} \times \mathbf{C}$. Whereas the second time derivative is

$$\partial_t^2 \boldsymbol{J}_{\mathbf{AC}} = \mu \boldsymbol{J}_{\mathbf{EH}} + \mu \partial_t^2 \mathbf{D} \times \mathbf{A} + \partial_t^2 \mathbf{B} \times \mathbf{C}. \tag{14}$$

This expression is the general relationship between helicity and chirality flow for arbitrary waves represented by real valued functions. For harmonic waves $\partial_t^2 \boldsymbol{J}_{\mathbf{AC}} = 0$, then

$$\mu \boldsymbol{J}_{\mathbf{EH}} + \mu \partial_t^2 \mathbf{D} \times \mathbf{A} + \partial_t^2 \mathbf{B} \times \mathbf{C} = \mu \boldsymbol{J}_{\mathbf{EH}} - \mu\varepsilon\omega^2 \mathbf{E} \times \mathbf{A} - \omega^2 \mathbf{B} \times \mathbf{C} = 0.$$

The relationship between chirality and helicity flow is then

$$\boldsymbol{J}_{\mathbf{EH}} = \frac{1}{\mu} \omega^2 \boldsymbol{J}_{\mathbf{AC}}. \tag{15}$$

4.4. Scheme

The chirality continuity equation has been obtained in section 2, via the complementary EM fields approach from the $\mathbf{E}$ and $\mathbf{B}$ inhomogeneous wave equations. The helicity continuity equation can also be obtained using the complementary fields approach, starting with the $\mathbf{A}$ and $\mathbf{C}$ inhomogeneous wave equations [34]. These routes are schematically depicted in figure 1. The potentials $\mathbf{A}$ and $\mathbf{C}$ wave equations are in turn obtained from the relationships between fields and potentials. Thus, depending on the starting point, whether the fields or the potentials wave equations, the complementary fields procedure yields the chirality or the helicity continuity equations. From the structure of these two equations, it is possible to transform directly from one to the other.

Rearrange the helicity continuity equation (12), to have similar form to the chirality equation

$$\nabla\cdot(-\mu\varepsilon\mathbf{A}\times\mathbf{E}-\mathbf{C}\times\mathbf{B})+\mu\varepsilon\partial_t\left(-\mathbf{C}\cdot\mathbf{E}+\mathbf{A}\cdot\mathbf{B}\right)=\mu\left(\int\mathbf{J}dt\cdot\mathbf{B}-\int\mathbf{B}dt\cdot\mathbf{J}\right).$$

Perform the mapping of all fields by replacing them with their derivatives, $\mathbf{A}\to\partial_t\mathbf{A}$, $\mathbf{C}\to\partial_t\mathbf{C}$, $\mathbf{E}\to\partial_t\mathbf{E}$, $\mathbf{B}\to\partial_t\mathbf{B}$, and $\mathbf{J}\to\partial_t\mathbf{J}$,

$$\begin{aligned}\nabla\cdot\left(-\mu\varepsilon\partial_t\mathbf{A}\times\partial_t\mathbf{E}-\partial_t\mathbf{C}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(-\partial_t\mathbf{C}\cdot\partial_t\mathbf{E}+\partial_t\mathbf{A}\cdot\partial_t\mathbf{B}\right)&\\=\mu\left(\int\partial_t\mathbf{J}dt\cdot\partial_t\mathbf{B}-\int\partial_t\mathbf{B}dt\cdot\partial_t\mathbf{J}\right)&.\end{aligned}$$

Substitute $\mathbf{E}=-\partial_t\mathbf{A}$ and $\mathbf{H}=-\frac{1}{\mu}\partial_t\mathbf{C}$ or $\mathbf{B}=-\partial_t\mathbf{C}$,

$$\nabla\cdot\left(\mu\varepsilon\mathbf{E}\times\partial_t\mathbf{E}+\mathbf{B}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\mathbf{B}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{B}\right)=\mu\left(\mathbf{J}\cdot\partial_t\mathbf{B}-\mathbf{B}\cdot\partial_t\mathbf{J}\right).$$

This equation is identical to (6d). Therefore, the chirality continuity equation is obtained from the helicity continuity equation via the mapping of all the fields by their time derivatives. The inverse mapping requires the integration of the fields. This scheme is shown in figure 1. This procedure could be repeated with chirality and its flow defined with curls as in (7b) and (7c) respectively, there is however a caveat. This curl mapping has been used before to transform the infra-zilch onto the zilch tensor [35]. The helicity conservation equation in terms of the curl of the potentials is

$$\begin{aligned}\nabla\cdot\left(\mathbf{A}\times(\nabla\times\mathbf{C})-\mathbf{C}\times\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\frac{1}{\mu\varepsilon}\mathbf{C}\cdot(\nabla\times\mathbf{C})+\mathbf{A}\cdot\mathbf{B}\right)&\\=\mu\mathbf{A}\cdot\int\nabla\times\mathbf{J}dt+\mu\mathbf{B}\cdot\int\mathbf{J}dt.&\qquad(16)\end{aligned}$$

Notice that it is not possible write the curl of the potential $\mathbf{C}$ in terms of the electric field without involving the current because $\nabla\times\mathbf{C}=-\left(\mu\varepsilon\mathbf{E}+\mu\int\mathbf{J}dt\right)$; That is, the curl of the $\mathbf{C}$ potential involves the sum of the electric field plus the (integral of the) current. Here again, this suggests that the curl not only involves the chirality of the field but also part of the rotational content of the sources. If the substitution were made, and currents grouped on the RHS, the helicity continuity equation (6d) is recovered. Nonetheless, it is possible to proceed leaving the continuity equation in terms of the potentials. Map the field functions in (16) onto their temporal derivatives, $\mathbf{A}\to\partial_t\mathbf{A}$, $\mathbf{C}\to\partial_t\mathbf{C}$, $\mathbf{B}\to\partial_t\mathbf{B}$, and $\mathbf{J}\to\partial_t\mathbf{J}$,

$$\begin{aligned}\nabla\cdot\left(\partial_t\mathbf{A}\times(\nabla\times\partial_t\mathbf{C})-\partial_t\mathbf{C}\times\partial_t\mathbf{B}\right)+\mu\varepsilon\partial_t\left(\frac{1}{\mu\varepsilon}\partial_t\mathbf{C}\cdot(\nabla\times\partial_t\mathbf{C})+\partial_t\mathbf{A}\cdot\partial_t\mathbf{B}\right)&\\=\mu\partial_t\mathbf{A}\cdot\int\nabla\times\partial_t\mathbf{J}dt+\mu\partial_t\mathbf{B}\cdot\int\partial_t\mathbf{J}dt.&\end{aligned}$$

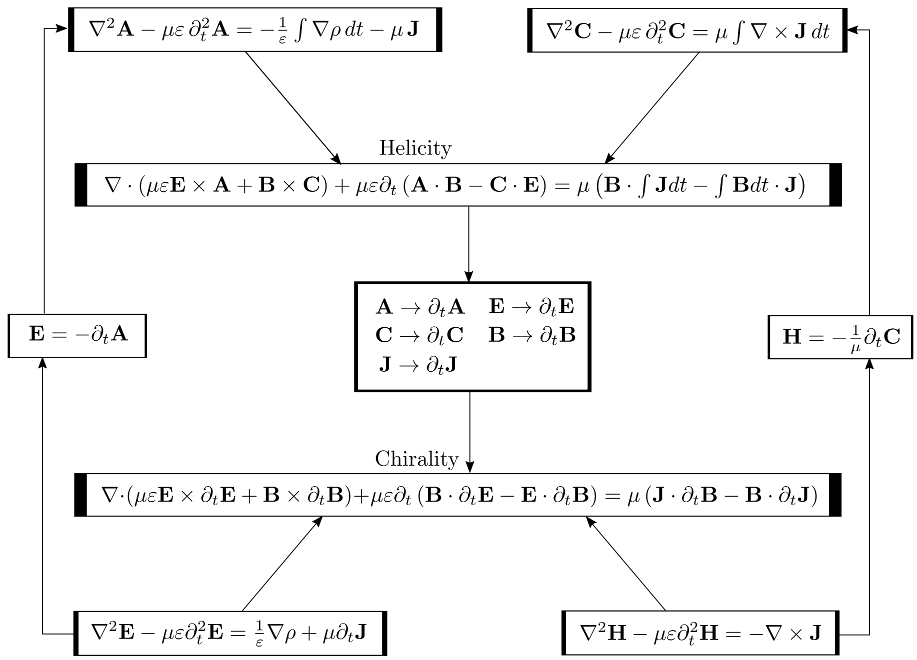

Figure 1: Helicity - Chirality continuity equations. Helicity is obtained via the $\mathbf{A}$ and $\mathbf{C}$ complementary fields, whereas chirality is obtained via the complementary electromagnetic $\mathbf{E}$ and $\mathbf{B}$ fields. The chirality continuity equation is obtained from the helicity continuity equation via the time derivative mapping of all fields. The inverse mapping requires integration of the fields.

Replace the potentials in terms of the EM fields, $\mathbf{E} = -\partial_t\mathbf{A}$ and $\mathbf{H} = -\frac{1}{\mu}\partial_t\mathbf{C}$ or $\mathbf{B} = -\partial_t\mathbf{C}$, and from the Faraday induction equation $\partial_t\mathbf{B} = -\nabla\times\mathbf{E}$,

$$\nabla\cdot\left(\mathbf{E}\times(\nabla\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{E})\right) + \mu\varepsilon\partial_t\left(\frac{1}{\mu\varepsilon}\mathbf{B}\cdot(\nabla\times\mathbf{B}) + \mathbf{E}\cdot(\nabla\times\mathbf{E})\right)$$
$$= -\mu\mathbf{E}\cdot\nabla\times\mathbf{J} - \mu\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{J}.$$

This expression is equal (upon substitution of $\mathbf{B} = \mu\mathbf{H}$) to (7a), the continuity equation with chirality and flow defined in terms of the curls. These mappings can also be used to transform from the set of equations for the fields (Maxwell's equations) to their counterpart set for the potentials [40].

4.5. Units

The mechanical angular momentum units are Joule second, $[\mathbf{L}] = [\mathbf{r}\times\mathbf{p}] = \left[kg\,m^2\,s^{-1}\right] = [J\,s]$. The mechanical AM density is the AM per unit volume, its units are then $\left[J\,s\,m^{-3}\right]$. The flow of mechanical AM density is the preceding quantity times the propagation velocity, its units are $\left[J\,m^{-2}\right]$. The helicity flow $\boldsymbol{J}_{\mathbf{AC}} = \mu\varepsilon\mathbf{E}\times\mathbf{A} + \mathbf{B}\times\mathbf{C}$, has units $\left[V^2 m^{-4} s^3\right]$. If this quantity is multiplied by

$\frac{c}{\mu}$, that represents the square of the speed of light over the vacuum impedance, it then has the units of the flow of angular momentum density

$$\boldsymbol{S}_{\mathbf{AC}} = \frac{c}{\mu}\boldsymbol{J}_{\mathbf{AC}} = \frac{c}{\mu}\left(\mu\varepsilon\mathbf{E}\times\mathbf{A}+\mathbf{B}\times\mathbf{C}\right) = \left[J\,m^{-2}\right].$$

The corresponding helicity with angular momentum density units is

$$\mathcal{H}_{\mathbf{AC}} = \frac{c}{\mu}\varrho_{\mathbf{AC}} = c\varepsilon\left(\mathbf{A}\cdot\mathbf{B}-\mathbf{C}\cdot\mathbf{E}\right) = \left[J\,s\,m^{-3}\right].$$

One of the reasons given in the literature in favour of the helicity versus chirality, has been that it can be scaled so as to endow it with mechanical AM units. We have deliberately not scaled $\varrho_{\mathbf{AC}}$ nor $\boldsymbol{J}_{\mathbf{AC}}$ before in the text in order to keep the units that arise from the wave equations. If the chirality (6b) and its flow (6c) are multiplied by the speed of light,

$$\frac{1}{\sqrt{\mu\varepsilon}}\varrho_{\mathbf{EH}} = \sqrt{\mu\varepsilon}\left(\mathbf{H}\cdot\partial_t\mathbf{E}-\mathbf{E}\cdot\partial_t\mathbf{H}\right) = \left[J\,m^{-3}\,s^{-1}\right] = \left[J\,s\,m^{-3}\,s^{-2}\right],$$

$$\frac{1}{\sqrt{\mu\varepsilon}}\boldsymbol{J}_{\mathbf{EH}} = \frac{1}{\sqrt{\mu\varepsilon}}\left(\mathbf{E}\times\partial_t\mathbf{D}+\mathbf{H}\times\partial_t\mathbf{B}\right) = \left[Jm^{-2}s^{-2}\right].$$

The quantities $\frac{1}{\sqrt{\mu\varepsilon}}\varrho_{\mathbf{EH}}$ and $\frac{1}{\sqrt{\mu\varepsilon}}\boldsymbol{J}_{\mathbf{EH}}$ then have units of angular momentum density and flow rationalized by squared time. This quadratic time factor comes from the mapping of the bilinear field functions onto their temporal partial derivatives.

5. Final remarks

The chirality continuity equation with sources has been derived using the complementary fields formalism. Two possible definitions of chirality and its flow have been compared with previous results. The fact that an averaging process need not be enforced has been physically explained in terms of the exchange between two fields, referred to as the complementary fields. The elusive general relationship between helicity and chirality has been obtained for arbitrary wave fields.

Since the helicity has been associated with mechanical angular momentum in subsection 4.5, the question naturally arises regarding the mechanical concept associated with chirality. Angular momentum in classical mechanics is defined by $\mathbf{L} = \boldsymbol{J}_{\mathbf{rp}} \equiv \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\dot{\mathbf{r}}$, where $\mathbf{r}$ is position and $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$, the linear momentum. Overhead dots represent time derivatives. The derivative of the AM is the torque $\boldsymbol{\tau} = \partial_t\mathbf{L} = \mathbf{r}\times\dot{\mathbf{p}} = \mathbf{r}\times\mathbf{F}$. The second derivative of the AM is

$$\partial_t^2\mathbf{L} = \dot{\mathbf{r}}\times\dot{\mathbf{p}}+\mathbf{r}\times\ddot{\mathbf{p}}.$$

Define the cross product of velocity times force as a measure of the rotational content,

$$\boldsymbol{J}_{\mathbf{vF}} \equiv \dot{\mathbf{r}}\times\dot{\mathbf{p}} = \mathbf{v}\times\mathbf{F}. \tag{17}$$

Then

$$\partial_t^2 \boldsymbol{J}_{\mathbf{rp}} = \boldsymbol{J}_{\mathbf{vF}} + \mathbf{r} \times \ddot{\mathbf{p}}.$$

$\boldsymbol{J}_{\mathbf{rp}}$ and $\boldsymbol{J}_{\mathbf{vF}}$ are both time odd pseudo vectors, cross product of two polar vectors. $\boldsymbol{J}_{\mathbf{rp}}$ is dependent on where the observer places the origin but $\boldsymbol{J}_{\mathbf{vF}}$ is origin independent. Both vectors are related to the rotational content of the system. Indeed, as the reader suspects, $\boldsymbol{J}_{\mathbf{rp}}$ corresponds to the electromagnetic helicity flow whereas $\boldsymbol{J}_{\mathbf{vF}}$ corresponds to the chirality flow. Due to the tiered structure of Maxwell's equations when written in terms of the fields, the potentials, the potentials of the potentials, etc. [40], the association can be stated at any two contiguous levels of each set of equations. For uniform circular motion in the $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y$ plane at angular frequency ω and radius ρ, the position vector is $\mathbf{r} = \rho \cos(\omega t)\, \hat{\mathbf{e}}_x + \rho \sin(\omega t)\, \hat{\mathbf{e}}_y$. From this expression, $|\boldsymbol{J}_{\mathbf{vF}}| = |\mathbf{v} \times \mathbf{F}| = m\rho^2\omega^3$ and the angular momentum is $|\boldsymbol{J}_{\mathbf{rp}}| = |\mathbf{r} \times \mathbf{p}| = m\rho^2\omega$. Their quotient is ω^2, analogous to the chirality over helicity quotient given by (15). The relationship between the second derivative of $\boldsymbol{J}_{\mathbf{rp}}$ and $\boldsymbol{J}_{\mathbf{vF}}$ is surprisingly similar to the relationship between helicity and chirality flow (14). Furthermore, the mapping that transforms helicity onto chirality is the same to the transformation from AM $\boldsymbol{J}_{\mathbf{rp}}$ to $\boldsymbol{J}_{\mathbf{vF}}$, namely, each variable is mapped into its time derivative.

The helicity (6b) and its flow (6c) defined in terms of time derivatives, were favoured against the definitions (7b), (7c) involving the curls. The relationship between them (8a), (8b), suggest that the curls definitions involve part of the source terms. Another argument outlined in the previous section, is that the curls in the helicity continuity equation cannot be written in terms of the fields without involving current terms. A further element in support of the time derivatives definitions is given from the analogy with mechanical momenta described above. As a final reflection along these lines, chirality may be thought of as the correlation of the field at a given time and at a later time. A measure of the twisting is given by the direction at which the (force) field points at a given time and its direction an instant afterwards. Take for example, the first term in the chirality flow $\varepsilon \mathbf{E} \times \partial_t \mathbf{E}$, from the definition of the derivative,

$$\varepsilon \mathbf{E} \times \partial_t \mathbf{E} = \varepsilon \mathbf{E}(t) \times \lim_{\delta t \to 0} \frac{\mathbf{E}(t + \delta t) - \mathbf{E}(t)}{\delta t}.$$

Distribution of the cross product within the limit gives

$$\mathbf{E} \times \partial_t \mathbf{E} \to \lim_{\delta t \to 0} \frac{\mathbf{E}(t) \times \mathbf{E}(t + \delta t)}{\delta t}. \tag{18}$$

Thus, if the field vector points in the same direction at different subsequent times this quantity is zero, whereas if the field vector changes direction, i.e. rotates, this quantity is a measure of its twisting. An analogous but apparently slightly more restrictive result, can be obtained for the scalar quantities related to rotation. Consider terms of the form $\mathbf{E} \cdot \partial_t \mathbf{H}$,

$$\mathbf{E} \cdot \partial_t \mathbf{H} = \lim_{\delta t \to 0} \frac{\mathbf{E}(t) \cdot \mathbf{H}(t + \delta t) - \mathbf{E}(t) \cdot \mathbf{H}(t)}{\delta t},$$

Provided that the Ampère-Maxwell equation (1c), can be integrated to $\mathbf{H} = \frac{1}{\mu c}\hat{\mathbf{k}} \times \mathbf{E}$, then $\mathbf{E}\left(t\right) \cdot \mathbf{H}\left(t\right) = 0$. The expression

$$\mathbf{E} \cdot \partial_t \mathbf{H} = \lim_{\delta t \to 0} \frac{\mathbf{E}\left(t\right) \cdot \mathbf{H}\left(t + \delta t\right)}{\delta t}, \tag{19}$$

is then a scalar measure of the rotation of the field $\mathbf{H}$ with respect to the field $\mathbf{E}$ at an infinitesimal later time. Therefore, in order to assess the rotational content of vector wavefields, be it via chirality, helicity or similar quantities, it is necessary to look for a temporal change in the angle of propagation of the vector wave fields.

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