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Ermakov-Lewis Invariant for Two Coupled Oscillators

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Abstract. We show that two coupled time dependent harmonic oscillators with equal frequencies have an invariant that is a generalization of the Ermakov-Lewis invariant for the single time dependent harmonic oscillator.

1. Introduction

Time dependent harmonic oscillators arise in several branches of physics, from classical mechanics to quantum mechanical systems such as optical trapping of atoms and molecules [1, 2, 3, 4].

The wavefunction of time independent harmonic oscillators has been reconstructed in ion-laser interactions [5] and quantized fields in cavities [6], and ways to overcome the effects of harming environments, that cause decoherence effects and therefore the destruction of nonclassical states of quantum systems, have been put forward [7, 8].

The extensions from single harmonic oscillators to coupled time dependent harmonic oscillators may be found in ion-laser interactions [1, 2, 3], quantized fields propagating through dielectric media [9], shortcuts to adiabaticity [10], the Casimir effect [11] to name some.

Constants of motion are of central importance in the study of dynamical systems. In particular, invariants in mechanical systems for time dependent Hamiltonian have attracted considerable interest over the years [12] and, in particular, time dependent harmonic oscillators (TDHO) have attracted attention due to their applications in several areas of physics [13, 14].

The most relevant cases are the linear potential [15], that may be produced in classical optic [16], and the quadratic spatial dependence that leads to the quantum mechanical time dependent harmonic oscillator (QM-TDHO) [17].

On the other hand, the simple extension to two coupled time dependent harmonic oscillators has been considered and its solution presented by Macedo and Guedes for a very limited case of time dependent functions [18]. This approach has been improved by the use of transformations that use orthogonal functions invariants [17] that simplifies the coupled harmonic oscillators Hamiltonian [19].

The QM-TDHO has been solved under various scenarios such as time dependent mass [20, 21] and applications of invariant methods have been used in adiabatic regimes [22] for the control of quantum noise [23] and the propagation of light in waveguide arrays [24, 25, 26, 27, 28, 29].



Moreover, the time evolution of time dependent harmonic oscillators may be treated via complex nonlinear Riccati equations [30, 31].

The main purpose of the present contribution is to obtain a generalization of the single harmonic oscillator Ermakov-Lewis invariant [32] to the case of two coupled harmonic oscillators, when the frequencies associated to the individual oscillators are equal. Although Thylwe and Korsche [33] have given a 'Ermakov-Lewis invariant' for the case of N coupled oscillators [34], we give here an invariant, for the two oscillators case, that has the same form as the one introduced by Lewis [32], and therefore, an auxiliary function that obeys Ermakov equation is needed in the invariant.

2. Lewis invariant

In the sixties, Lewis introduced a quantity called the invariant of the form (throughout the manuscript we will set $\hbar = 1$)

$$\hat{I}_x = \frac{1}{2} \left(\frac{\hat{x}^2}{\rho_x^2} + [\rho_x \hat{p}_x - \dot{\rho}_x \hat{x}]^2 \right), \quad (1)$$

with ρ_x an auxiliary function that obeys the Ermakov equation

$$\ddot{\rho}_x + \Omega_x^2(t) \rho_x = \frac{1}{\rho_x^3}, \quad (2)$$

and that, consequently, takes the name Ermakov-Lewis invariant. The operator given in (1) is invariant in the sense that

$$\frac{\partial \hat{I}_x}{\partial t} + i[\hat{I}_x, \hat{H}_x] = 0, \quad (3)$$

with \hat{H}_x the Hamiltonian for a (one-dimensional) single time dependent harmonic oscillator

$$\hat{H}_x(t) = \frac{1}{2} [\hat{p}_x^2 + \Omega_x^2(t) \hat{x}^2]. \quad (4)$$

We may note in the invariant (1) two main ingredients: a displacement in momentum by the position operator and the amplification or de-amplification of position or momentum operators, *i.e.*, squeezing [35, 36, 37, 38]. These ingredients, converted into unitary transformations, give

$$\hat{D}_f = e^{-\frac{i}{2} f(t) \hat{x}^2}, \quad \hat{S}_g = e^{i \frac{g(t)}{2} (\hat{x} \hat{p}_x + \hat{p}_x \hat{x})}, \quad (5)$$

where we used to write a solution to the TDHO Hamiltonian [17]. In the above equation, by taking $f(t) = \frac{\dot{\rho}_x}{\rho_x}$ and $g(t) = \ln \rho_x$, the time dependence in the Hamiltonian (4) may be factorized yielding an easy way to write the evolution operator for the transformed Hamiltonian. On the other hand, if the functions $f(t) = \frac{\dot{u}_x}{u_x}$ and $g(t) = \ln u_x$ are used, where the auxiliary function is solution of the classical equation of motion

$$\ddot{u}_x + \Omega_x^2(t) u_x = 0, \quad (6)$$

the term proportional to \hat{x}^2 is eliminated from the Hamiltonian, *i.e.*, yielding a free particle [17].

A relation between the auxiliary functions that simplify the Hamiltonian (4) may be obtained as

$$u_x = \rho_x \cos \left(\int \frac{d\tau}{\rho_x^2} \right), \quad \rho_x = u_x \sqrt{1 + \left(\int \frac{d\tau}{u_x^2} \right)^2}, \quad (7)$$

such that, once we may solve equations (2) or (6), we may find the other auxiliary function.

3. Coupled time dependent harmonic oscillators

We consider the time-dependent Hamiltonian for two interacting harmonic oscillators (we set the masses equal to one)

$$\hat{H}(t) = \frac{1}{2} [\hat{p}_x^2 + \hat{p}_y^2 + \kappa_x(t)\hat{x}^2 + \kappa_y(t)\hat{y}^2] + \kappa(t)(\hat{x} - \hat{y})^2, \quad (8)$$

with κ 's being the time-dependent spring parameters. By setting

$$\Omega_x^2(t) = \kappa_x(t) + \kappa(t), \quad (9)$$

and

$$\Omega_y^2(t) = \kappa_y(t) + \kappa(t) \quad (10)$$

and $\eta(t) = -2\kappa(t)$, such that we end up with a Hamiltonian of the form

$$\hat{H}_{xy}(t) = \frac{1}{2} [\hat{p}_x^2 + \hat{p}_y^2 + \Omega_x^2(t)\hat{x}^2 + \Omega_y^2(t)\hat{y}^2] + \eta(t)\hat{x}\hat{y}, \quad (11)$$

whose classical equations of motion are

$$\ddot{u}_x + \Omega_x^2(t)u_x = -\eta(t)u_y, \quad \ddot{u}_y + \Omega_y^2(t)u_y = -\eta(t)u_x. \quad (12)$$

If we consider the case of equal frequencies $\Omega_x(t) = \Omega_y(t) = \Omega(t)$, the Hamiltonian above reduces to

$$\hat{H}_{xy} = \frac{1}{2} [\hat{p}_x^2 + \hat{p}_y^2 + \Omega^2(t)\hat{x}^2 + \Omega^2(t)\hat{y}^2 + 2\eta(t)\hat{x}\hat{y}]. \quad (13)$$

It is not difficult to show that for this Hamiltonian it exists an invariant of the form

$$\hat{I}_{xy} = \frac{(\hat{x} + \hat{y})^2}{\rho^2} + [\rho(\hat{p}_x + \hat{p}_y) - \dot{\rho}(\hat{x} + \hat{y})]^2, \quad (14)$$

because

$$[\hat{I}_{xy}, \hat{H}_{xy}] = i \left\{ \frac{1}{\rho^2} + \dot{\rho}^2 - [\Omega^2(t) + \eta(t)]\rho^2 \right\} [(\hat{x} + \hat{y})(\hat{p}_x + \hat{p}_y) + (\hat{p}_x + \hat{p}_y)(\hat{x} + \hat{y})] \\ - 2i\rho\dot{\rho}(\hat{p}_x + \hat{p}_y)^2 + 2i\rho\dot{\rho}(\hat{x} + \hat{y})^2[\Omega^2(t) + \eta(t)], \quad (15)$$

and

$$\frac{\partial \hat{I}_{xy}}{\partial t} = -(\rho\ddot{\rho} + \dot{\rho}^2)[(\hat{x} + \hat{y})(\hat{p}_x + \hat{p}_y) + (\hat{p}_x + \hat{p}_y)(\hat{x} + \hat{y})] \\ + 2\rho\dot{\rho}(\hat{p}_x + \hat{p}_y)^2 + 2\left(\rho\dot{\rho} - \frac{\dot{\rho}}{\rho^3}\right)(\hat{x} + \hat{y})^2, \quad (16)$$

such that

$$\frac{\partial \hat{I}_{xy}}{\partial t} + [\hat{I}_{xy}, \hat{H}_{xy}] = 0. \quad (17)$$

The invariant (14) has the same structure of the Ermakov-Lewis invariant (1) for the single time dependent harmonic oscillator. It is easy to prove that the auxiliary function, ρ , obeys the Ermakov equation

$$\ddot{\rho} + [\Omega^2(t) + \eta(t)]\rho = \frac{1}{\rho^3}. \quad (18)$$

4. Solution for equal frequencies

We now consider the time-dependent transformation [17]

$$\hat{T}_v = e^{i\frac{\ln v(t)}{2}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})} e^{-i\frac{\dot{v}(t)}{2v(t)}\hat{x}^2} e^{i\frac{\ln v(t)}{2}(\hat{y}\hat{p}_y + \hat{p}_y\hat{y})} e^{-i\frac{\dot{v}(t)}{2v(t)}\hat{y}^2} \quad (19)$$

that produces the set of transformed quantities

$$\begin{aligned} \hat{T}_v \hat{x} \hat{T}_v^\dagger &= v\hat{x} \\ \hat{T}_v \hat{y} \hat{T}_v^\dagger &= v\hat{y} \\ \hat{T}_v \hat{p}_x \hat{T}_v^\dagger &= \frac{\hat{p}_x}{v} + \dot{v}\hat{x}, \\ \hat{T}_v \hat{p}_y \hat{T}_v^\dagger &= \frac{\hat{p}_y}{v} + \dot{v}\hat{y}, \end{aligned} \quad (20)$$

where v is the solution of the equation

$$\ddot{v} + [\Omega^2(t) + \eta(t)]v = 0. \quad (21)$$

If we transform the wave function with the transformation above, i.e.,

$$|\phi_v(t)\rangle = \hat{T}_v |\psi(t)\rangle, \quad (22)$$

the Schrödinger equation

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)|\psi(t)\rangle \quad (23)$$

needs to be transformed. We do it by substituting (22) into (23) which gives

$$i\left(\hat{T}_v^\dagger \frac{\partial|\phi_v(t)\rangle}{\partial t} + \frac{\partial\hat{T}_v^\dagger}{\partial t} |\phi_v(t)\rangle\right) = \hat{H}(t)\hat{T}_v^\dagger |\phi_v(t)\rangle. \quad (24)$$

By noting that

$$\frac{\partial\hat{T}_v^\dagger}{\partial t} = \frac{i}{2}\hat{T}_v^\dagger \left[(v\ddot{v} - \dot{v}^2)\hat{x}^2 - \frac{\dot{v}}{v}(\hat{p}_x\hat{x} + \hat{x}\hat{p}_x) + (v\ddot{v} - \dot{v}^2)\hat{y}^2 - \frac{\dot{v}}{v}(\hat{p}_y\hat{y} + \hat{y}\hat{p}_y) \right], \quad (25)$$

such that

$$i\frac{\partial|\phi_v(t)\rangle}{\partial t} = \frac{1}{2} \left[\frac{\hat{p}_x^2}{v^2} + (\Omega^2 v + \ddot{v})v\hat{x}^2 + \frac{\hat{p}_y^2}{v^2} + (\Omega^2 v + \ddot{v})v\hat{y}^2 + 2\eta(t)v^2\hat{x}\hat{y} \right] |\phi_v(t)\rangle, \quad (26)$$

from (21), we may rewrite the Schrödinger equation as

$$i\frac{\partial|\phi_v(t)\rangle}{\partial t} = \frac{1}{2} \left[\frac{\hat{p}_x^2}{v^2} + \frac{\hat{p}_y^2}{v^2} - \eta(t)v^2(\hat{x} - \hat{y})^2 \right] |\phi_v(t)\rangle. \quad (27)$$

Now, performing a second transformation, $|\phi_\theta\rangle = \hat{R}_\theta |\phi_u\rangle$, with $\hat{R}_\theta = \exp[i\theta(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)]$

$$i\frac{\partial|\phi_\theta\rangle}{\partial t} = \hat{H}_\theta(t)|\phi_\theta\rangle \quad (28)$$

which transform the relevant operators according to

$$\begin{aligned}
\hat{R}_\theta \hat{x} \hat{R}_\theta^\dagger &= \hat{x} \cos \theta - \hat{y} \sin \theta, \\
\hat{R}_\theta \hat{y} \hat{R}_\theta^\dagger &= \hat{y} \cos \theta + \hat{x} \sin \theta, \\
\hat{R}_\theta \hat{p}_x \hat{R}_\theta^\dagger &= \hat{p}_x \cos \theta - \hat{p}_y \sin \theta, \\
\hat{R}_\theta \hat{p}_y \hat{R}_\theta^\dagger &= \hat{p}_y \cos \theta + \hat{p}_x \sin \theta,
\end{aligned} \tag{29}$$

and by setting $\theta = \pi/4$ we obtain the integrable equation

$$i \frac{\partial |\phi_\theta(t)\rangle}{\partial t} = \left[\frac{\hat{p}_x^2 + \hat{p}_y^2}{2v^2(t)} - \eta(t)v^2\hat{y}^2 \right] |\phi_\theta(t)\rangle. \tag{30}$$

In the above equation we may see that we have a time dependent harmonic oscillator Hamiltonian in the variable y , while a free particle in the variable x , whose solutions are well known [39].

5. Conclusions

We have shown that the Ermakov-Lewis invariant for the (one-dimensional) single time dependent harmonic oscillator can be generalized to the two-coupled harmonic oscillators case when equal time dependent frequencies are considered. The Invariant keeps the same structure as well as the Ermakov equation that needs to be solved, replacing the time dependent frequency, $\Omega^2(t)$, by an effective frequency, $\Omega^2(t) + \eta(t)$.

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References

- [1] Moya-Cessa H and Tombesi P 2000 *Phys. Rev. A* **61** 025401
- [2] Moya-Cessa H, Jonathan D and Knight P L 2003 *J. Mod. Opt.* **50** 265
- [3] Casanova J, Puebla R, Moya-Cessa H and Plenio M B 2018 *npj Quant. Inf.* **5** 47
- [4] Cronin A D, Schmiedmayer J and Pritchard D E 2009 *Rev. Mod. Phys.* **81** 1051
- [5] Leibfried D, Meekhof D M, King B E, Monroe C, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **77** 4281
- [6] Bertet P, Auffeves A, Maioli P, Osnaghi S, Meunier T, Brune M, Raimond J M and Haroche S 2002 *Phys. Rev. Lett.* **89** 200402
- [7] Moya-Cessa H, Roversi J A, Dutra S M and Vidiella-Barranco A 1999 *Phys. Rev. A* **60** 4029
- [8] Moya-Cessa H, Dutra S M, Roversi J A and Vidiella-Barranco A 1999 *J. Mod. Opt.* **46** 555
- [9] Dodonov V V and Klimov A B 1996 *Phys. Rev. A* **53** 2664
- [10] Chen X, Ruschhaupt A, Schmidt S, del Campo A, Guéry-Odelin D and Muga J G 2010 *Phys. Rev. Lett.* **104** 063002
- [11] Román-Ancheyta R, Ramos-Prieto I, Perez-Leija A, Busch K and León-Montiel R D J *Phys. Rev. A* 2017 **96** 032501
- [12] Bouquet S and Lewis V J 1996 *J. Math. Phys.* **37** 5509
- [13] Colegrave R K and Mannan M A 1988 *J. Math. Phys.* **29** 1580
- [14] Fernández Guasti M and Moya-Cessa H 2003 *Phys. Rev. A* **67** 063803
- [15] Guedes I 2001 *Phys. Rev. A* **63** 034102
- [16] Chávez-Cerda S, Ruiz-Corona U, Arrizon V and Moya-Cessa H M 2011 *Opt. Express* **19** 16448
- [17] Fernández Guasti M and Moya-Cessa H 2003 *J. Phys. A* **36** 2069
- [18] Macedo D X and Guedes I 2012 *J. Math. Phys.* **53** 052101
- [19] Urzúa A R, Ramos-Prieto I, Fernández-Guasti M and Moya-Cessa H M 2019 *Quant. Rep.* **1** 82
- [20] Moya-Cessa H and Fernández Guasti M 2007 *Rev. Mex. Fis.* **53** 42
- [21] Ramos-Prieto I, Espinosa-Zuñiga A, Fernández-Guasti M and Moya-Cessa H M 2018 *Mod. Phys. Lett. B* **32** 1850235

- [22] de Ponte M A and Santos A C 2018 *Quant. Inf. Proc.* **17** 149
- [23] Levy A, Kiely A, Muga J G, Kosloff R and Torrontegui E 2018 *New J. Phys.* **20** 025006
- [24] Barral D and Liñares J 2016 *Opt. Comm.* **359** 61
- [25] Barral D and Liñares J 2015 *J. Opt. Soc. Am. B* **32** 1993
- [26] Perez-Leija A, Keil R, Moya-Cessa H, Szameit A and Christodoulides D N 2013 *Phys. Rev. A* **87** 022303
- [27] Perez-Leija A, Hernandez-Herrejon J C, Moya-Cessa H, Szameit A and Christodoulides D N 2013 *Phys. Rev. A* **87** 013842
- [28] Keil R, Perez-Leija A, Aleahmad A, Moya-Cessa H, Christodoulides D N and Szameit A 2012 *Opt. Lett.* **37** 3801
- [29] Rodriguez-Lara B M, Zarate-Cardenas A, Soto-Eguibar F and Moya-Cessa H M 2013 *Opt. Express* **21** 12888
- [30] Cruz H, Schuch D, Castanos O and Rosas-Ortiz O 2015 *Ann. of Phys.* **360** 44
- [31] Cruz H, Schuch D, Castanos O and Rosas-Ortiz O 2016 *Ann. of Phys.* **373** 609
- [32] Lewis H R 1967 *Phys. Rev. Lett.* **18** 510
- [33] Thylwe K E and Korsch H J 1998 *J. Phys. A: Math. Gen.* **31** L279
- [34] Leon-Montiel R J, Quiroz-Juarez M A, Quintero-Torres R, Dominguez-Juarez J L, Moya-Cessa H M, Torres J P and Aragon J L 2015 *Sc. Rep.* **5** 17339
- [35] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
- [36] Yuen H P 1976 *Phys. Rev. A* **13** 2226
- [37] Caves C M 1981 *Phys. Rev. D* **23** 1693
- [38] Moya-Cessa H M and Vidiella-Barranco A 1992 *J. Mod. Opt.* **39** 2481
- [39] Pedrosa I A 1997 *Phys. Rev. A* **55** 3219