Research Article

Energy density and flow distribution in a vibrating string

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Abstract
The kinetic and potential energy of a vibrating string is considered in the first-order approximation of purely transverse small amplitude linear oscillations. The energy continuity equation is obtained for an energy density having a potential that depends on second-order spatial derivatives of the perturbation. The concomitant flow involves two terms $-\partial_t\psi\partial_z\psi$ and $\psi\partial_z\partial_t\psi$ that will be shown to correspond to the kinetic and potential energy flows respectively. The string's transverse force is consistent with its derivation from minus the gradient of this potential. In contrast, the widely accepted potential energy of a string depends on the perturbation's first-order derivative squared. However, this expression does not yield the appropriate force required to satisfy a wave equation. The potential energy controversy is thus resolved in favor of the $\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi$ potential function. Contrary to what is usually recognized, the potential energy spatial distribution is shown to be uniquely determined. These results have far-reaching consequences pertaining the wave energy distribution in other mechanical systems.

Keywords Energy transport · Continuity equations · Potential energy localization

1 Introduction

The rather complex problem of vibrations in a taut string [1] can be cast in its simplest form, considering only small transverse vibrations with constant tension. These approximations, upheld throughout this text, are the starting point in order to establish the main features of the physical phenomenon. This system is often presented in the introductory chapter to transverse wave phenomena in continuous media [2].

Contrary to what would be expected, there is still considerable controversy regarding the energy density and flow in this elementary problem [3]. This issue can be traced back to a venerable reference [4], where "the energy of only a portion of a string" is stated not to be unique. Nonetheless, it was also stressed from the outset that the energy density integrated over the complete string must be unique. The quadratic form of a first order derivative for the potential energy has been preferred to the form involving a second order derivative [5]. In [4], the first-order spatial derivative expression was chosen due to its simpler mathematical expression. However, it was later supported by a formal derivation of a continuity equation [6]. Whether the starting point, in order to describe the energy of a system, is an integral or a differential form, depends on the system and the information required. If the energy distribution is required, the differential form is essential, whereas if the energy evolution of the whole system is required, the integral form is more appropriate, in particular if discontinuities such as shock waves are present [7]. For these and other reasons that will be considered later, the square of the first-order derivative has prevailed as the quantity representing the potential energy of this system [2]. Nonetheless, it has also been argued, in particular regarding counter-propagating waves, that the form involving second-order derivatives provides a better description of the potential energy spatial distribution within a mode [8]. The potential energy density ambiguity

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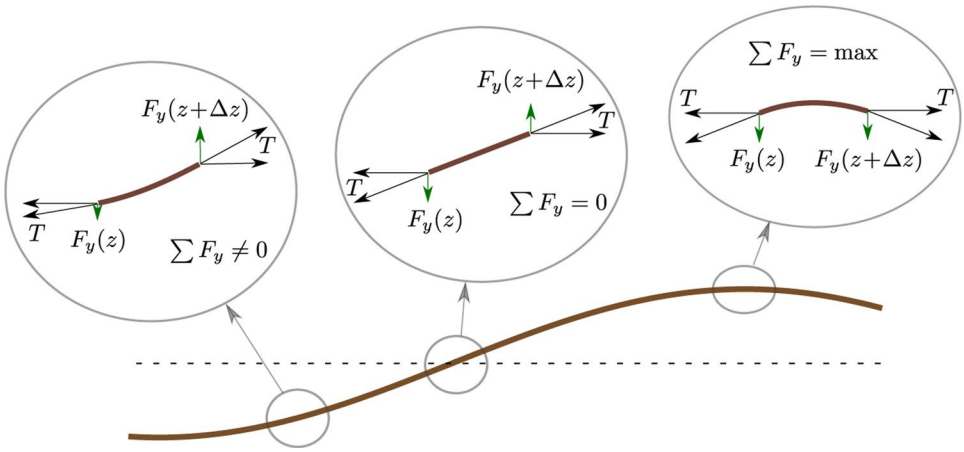

Fig. 1 Forces exerted on infinitesimal segments of the oscillating string. Leftmost segment is an arbitrary point between the trough and the equilibrium position; middle segment is located at the inflection point; right hand segment at the wave crest. The net force is proportional to the curvature of the string

has triggered the more ample discussion of where the potential energy resides [9]. The energy in mechanical wave phenomena is of great importance for extracting energy for human oriented purposes. The energy distribution in space and time is essential to develop highly efficient conversion devices. For example, in order to obtain energy from ocean waves or energy from vibrations generated in mechanical devices by wind energy.

In this manuscript, a continuity equation is obtained from the wave equation in Sect. 2. This rigorous derivation, puts the potential energy density involving second order derivatives, hereafter labeled by ϱ_{pot}, on an equal footing to the first order derivative expression. In Sect. 2.3, the infinite set of conservation equations that can be derived from the wave equation is addressed. Whether there is a way to choose a continuity equation from this set, to unequivocally represent energy conservation, will be treated in Sect. 3. The force will be derived from the proposed potential functions in Sects. 3.1 and 3.2. From these results, it will be shown in the last section that ϱ_{pot} represents the potential energy of the system. In the appendix A, the well known traveling harmonic, counter-propagating and waves superposition cases will be revisited in the light of the new results and contrasted with the first order derivative expressions.

2 Complementary fields conservation equation

The wave equation for a perturbation ψ in $1+1$ dimensions, one temporal labeled t and one spatial, labeled z, is

$$\partial_z^2\psi - \frac{1}{v^2}\partial_t^2\psi = 0, \tag{1}$$

where v is the perturbation's velocity of propagation, and ∂_z^2, ∂_t^2 represent second partial derivatives with respect to the z spatial propagation direction and time, respectively. For a string, the perturbation ψ is transverse and is actually the position of the string in an orthogonal direction, say the y direction. This differential equation is obtained from Newton's relationship, equating the acceleration to the force exerted on the two ends of an infinitesimal string segment. The force on various segments at different positions of the string is illustrated in Fig. 1. The transverse force F_y, at each end is equal to the tension force T, assumed to be constant, times the projection in the transverse direction. For small amplitudes, $\sin \approx \tan = \partial_z\psi$, the projection is equal to the string's slope. The transverse force linear density on a segment is,

$$F_y(\text{segment}) = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left(T\left(\partial_z\psi\right)_{z+\Delta z} - T\left(\partial_z\psi\right)_z\right) = T\partial_z^2\psi. \tag{2}$$

As pertinently asserted "*The tension therefore provides a net transverse force on the segment only when the string is curved*" [2, p. 142]. To state it formally, the curvature of a function ψ is $\kappa = \partial_z^2\psi\left(1+\left(\partial_z\psi\right)^2\right)^{-\frac{3}{2}}$. Therefore, the force on a segment is proportional to the curvature if the slope is small. The displacement ψ in the y direction, then satisfies the wave equation

$$\mu\partial_t^2\psi = T\partial_z^2\psi, \tag{3}$$

where μ is the mass linear density. From comparison with Eq. (1), the velocity of propagation is $v = \sqrt{\frac{T}{\mu}}$. If the solution is written in amplitude and phase variables, $\psi = A\cos(kz-\omega t)$, the velocity also obeys the relationship $v = \frac{\omega}{k}$, where ω is the angular frequency and k is the wavevector magnitude. Let us insist on what is evident,

the second spatial derivative is essential to obtain a wave equation, for a first-order derivative force would produce a parabolic equation.

The complementary fields scheme allows us to obtain a continuity equation from two linearly independent solutions to the scalar wave equation [10]. In the one-dimensional case, the conserved quantity becomes an exact invariant [11]. The complementary fields procedure can be generalized to any two equations from a set of second-order ordinary differential equations (ODE's). This scheme has been successfully applied to obtain the energy content of a time dependent harmonic oscillator subjected to a spatially linear but arbitrary time dependent restoring force [12]. The complementary fields formalism has also been used to obtain continuity equations from vector second-order linear partial differential equations (PDE's). In this way, the helicity gauge independent continuity equation of electromagnetic fields was recently derived [13]. From a physical point of view, the fields are complementary because the energy content of the system is dynamically exchanged between the two fields involved.

Evaluate the time derivative of the wave equation (1),

$$\partial_z^2\dot{\psi} - \frac{1}{v^2}\partial_t^2\dot{\psi} = 0, \tag{4}$$

where $\dot{\psi} = \partial_t\psi$. Take the product of the velocity function $\dot{\psi}$ times the wave equation (1),

$$\dot{\psi}\partial_z^2\psi - \frac{1}{v^2}\dot{\psi}\partial_t^2\psi = 0. \tag{5}$$

Evaluate the product of the perturbation function ψ times the derivative of wave equation (4),

$$\psi\partial_z^2\dot{\psi} - \frac{1}{v^2}\psi\partial_t^2\dot{\psi} = 0.$$

The difference between these two PDE's is

$$\left(\psi\partial_z^2\dot{\psi} - \dot{\psi}\partial_z^2\psi\right) - \frac{1}{v^2}\left(\psi\partial_t^2\dot{\psi} - \dot{\psi}\partial_t^2\psi\right) = 0.$$

The two terms in parenthesis can be written in terms of partial derivatives to obtain a continuity equation

$$\partial_z\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right) + \frac{1}{v^2}\partial_t\left(\left(\partial_t\psi\right)^2 - \psi\partial_t^2\psi\right) = 0,$$

where $\dot{\psi}$ has been explicitly written as $\partial_t\psi$. In this 1+1 dimensional case, the divergence operator is simply the partial derivative with respect to z. Invoke the wave equation to rewrite the term involving the second time derivative in terms of the spatial derivative

$$\partial_z\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right) + \frac{1}{v^2}\partial_t\left(\left(\partial_t\psi\right)^2 - v^2\psi\partial_z^2\psi\right) = 0.$$

Multiply the equation by the tension T and substitute $v^2 = \frac{T}{\mu}$, to obtain the string continuity equation

$$\partial_z\left[\frac{1}{2}T\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right)\right] + \partial_t\left[\frac{1}{2}\left(\mu\left(\partial_t\psi\right)^2 - T\psi\partial_z^2\psi\right)\right] = 0 \tag{6}$$

There is then an assessed quantity,

$$\varrho = \underbrace{\frac{1}{2}\mu\left(\partial_t\psi\right)^2}_{\text{kinetic energy}} + \underbrace{-\frac{1}{2}T\psi\partial_z^2\psi}_{\text{potential energy}}, \tag{7a}$$

where the differential equation (6) has been multiplied by $\frac{1}{2}$ to comply with the usual definition of kinetic energy density,

$$\varrho_{\text{kin}} = \frac{1}{2}\mu\left(\partial_t\psi\right)^2. \tag{7b}$$

The remaining term in the assessed density must then be a potential energy

$$\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi. \tag{7c}$$

At this stage, this assertion can be considered as a working hypothesis. The quantity ϱ is then the total energy density of the string. The corresponding flow is

$$\mathbf{\Phi}_\varrho = \frac{1}{2}T\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right). \tag{8}$$

The potential energy density (7c) has been obtained before using a force times distance argument of an infinitesimal string segment evaluated for a continuous quasi-static transition of the string [4, 8, p. 126]. However, only recently a continuity equation of the form (6) has been stated [9]. In order to elucidate the contributions to the flow, write the wave velocity in the restricted case of a positive direction traveling wave $v = -\frac{\partial_t\psi}{\partial_z\psi}$. The flow should be equal to the product of the energy density times the velocity

$$\varrho\, v = \left(\varrho_{\text{kin}}v + \varrho_{\text{pot}}v\right) = \frac{1}{2}\left(\mu\left(\partial_t\psi\right)^2 v - T\psi\partial_z^2\psi\, v\right).$$

From the wave velocity, $\partial_t\psi = -v\partial_z\psi = -\sqrt{\frac{T}{\mu}}\partial_z\psi$, substitute its square in the first term. From the spatial derivative of this expression $\partial_z\partial_t\psi = -v\partial_z^2\psi$, this result is substituted in the second term to obtain

$$\left(\varrho_{\text{kin}}v + \varrho_{\text{pot}}v\right) = \frac{1}{2}\left(-T\partial_z\psi\partial_t\psi + T\psi\partial_z\partial_t\psi\right),$$

that is identical to the flow (8) obtained from the complementary fields procedure. From this latter derivation, we identify the kinetic energy flow as

$$\mathbf{\Phi}_{\varrho\text{kin}} = \varrho_{\text{kin}} v = -\frac{1}{2} T \partial_z \psi \partial_t \psi, \tag{9a}$$

whereas the potential energy flow is

$$\mathbf{\Phi}_{\varrho\text{pot}} = \varrho_{\text{pot}} v = \frac{1}{2} T \psi \partial_z \partial_t \psi. \tag{9b}$$

2.1 Energy density with first order derivative

In contrast, the usual first order derivative expression uses only Eq. (5) and invokes the identity $\partial_t \psi \partial_z^2 \psi = \partial_z \left(\partial_t \psi \partial_z \psi \right) - \frac{1}{2} \partial_t \left(\partial_z \psi \right)^2$ to obtain the continuity equation [14],

$$\partial_z \left(-T \partial_t \psi \partial_z \psi \right) + \partial_t \frac{1}{2} \left(\mu \left(\partial_t \psi \right)^2 + T \left(\partial_z \psi \right)^2 \right) = 0. \tag{10}$$

The assessed density and flow are then

$$\mathcal{E}_1 = \underbrace{\frac{1}{2} \mu \left(\partial_t \psi \right)^2}_{\text{kinetic energy } \mathcal{E}_1^{\text{kin}}} + \underbrace{\frac{1}{2} T \left(\partial_z \psi \right)^2}_{\text{potential energy } V_1}, \quad P_1 = \underbrace{-T \partial_z \psi \partial_t \psi}_{\text{power}}, \tag{11}$$

the subindex 1 is used throughout this text to emphasize that these expressions correspond to the first order derivative expression. These energy and flow expressions have also been obtained by other methods [15], making this proposal more robust.

2.2 Comparison of the two approaches

The kinetic energy density is the same in both schemes. However, the potential energy density involves a second-order derivative in the complementary fields derivation, whereas in the usual expression it is proportional to the square of the first spatial derivative. The flow had been considered to be equal in previous proposals. However, the present derivation imposes a new expression for the flow, where the usual form only accounts for half the usual flow, the remaining term $\frac{1}{2} T \psi \partial_z \partial_t \psi$ being entirely new.

The strongest argument to dismiss ϱ_{pot} from a mathematical point of view, has been that it does not satisfy the energy transport equation [16, Eq. (7)],

$$\frac{\partial}{\partial z} \left(-T \partial_t \psi \partial_z \psi \right) + \partial_t \frac{1}{2} \left(\mu \left(\partial_t \psi \right)^2 - T \psi \partial_z^2 \psi \right) \neq 0.$$

Indeed, at the time this criticism was written, there was no suggestion that the flow had to be modified accordingly, leading to the flow definition given by (8). Therefore, with the appropriate flow expression, the energy transport equation (6), is indeed fulfilled.

It has been stated that the potential energy ϱ_{pot} is only valid for calculation of the potential energy of the entire string [3, 16]. The present derivation makes it clear that this is not the case. The differential form insures that there is local as well as global conservation.[1] The energy densities and its flow can be evaluated over an arbitrary segment, be it a sub-wavelength segment (so long as the medium can be considered continuous) or the entire string. In this sense, it stands on an equal footing to the first-order spatial derivative expression. Both procedures are equally sound and both possess differential forms. The assertion that ϱ_{pot} is only globally valid, has its origin in the way the two potential terms have been related [4]. If the ϱ_{pot} potential energy is integrated over the entire string and integration by parts is performed,

$$\int_a^b \varrho_{\text{pot}} \, \mathrm{d}z = -\int_a^b \frac{1}{2} T \psi \partial_z^2 \psi \, \mathrm{d}z = \frac{1}{2} T \int_a^b \left(\partial_z \psi \right)^2 \mathrm{d}z - \frac{1}{2} T \psi \partial_z \psi \Big|_a^b.$$

(A factor of $\frac{1}{2}$ due to a $\beta d\beta$ integration from 0 to 1 is not needed here [4]). From this equation, a relationship between the integral of the two potential energies ϱ_{pot} and V_1 is obtained

$$\int_a^b \varrho_{\text{pot}} \, \mathrm{d}z = \int_a^b V_1 \mathrm{d}z - \frac{1}{2} T \psi \partial_z \psi \Big|_a^b. \tag{12}$$

The spatial integrals of the two potential terms ϱ_{pot} and V_1 are then equal provided that the boundary terms add up to zero. According to [4, p. 127], if the limits are the ends of the string, whenever there is no energy transferred to the supports, either rigid or freely supported, the boundary term $-\frac{1}{2} T \psi \partial_z \psi \big|_a^b$ is zero. If the string is terminated by a damper or another string with characteristic impedance different from the string, energy will be transmitted (and reflected) at the interface [2, p. 151]. This state of affairs has laid an unduly importance to this boundary term. This argument has also been used to state that the potential energy function is not unique, i.e., ϱ_{pot} or V_1 are both admissible, but the energy of the whole, spatially integrated system, is indeed unique [4].

2.3 Infinite set of continuity equations

It has been known for some time that free fields and in particular electromagnetic fields, possess an infinite set of continuity equations [17, 18]. The densities are represented by local bilinear functions, the zilch tensors [19, 20] were early examples of these types of continuity equations. Our previous derivation allows for a straightforward demonstration of the existence of this infinite set. The complementary fields procedure requires two

[1] Recall that local conservation implies global conservation, but the implication does not hold the other way around.

second-order linear PDE's, let the complementary pair be

$$\partial_z^2\xi - v^{-2}\partial_t^2\xi = 0 \text{ and } \partial_z^2\chi - v^{-2}\partial_t^2\chi = 0.$$

Evaluating the nth order derivative of $\partial_z^2\psi - v^{-2}\partial_t^2\psi = 0$, the equation for $\xi = \partial_t^n\psi$ is obtained and the mth order derivative gives $\chi = \partial_t^m\psi$. The continuity equation will be nontrivial provided that the solutions ξ and χ are linearly independent functions. It is always possible to construct such independent solutions, for example, from the Wronskian of the system. The continuity equation, following our previous construction but with these two equations as the starting point is

$$\partial_z\left(\xi\partial_z\chi - \chi\partial_z\xi\right) + v^{-2}\partial_t\left(\chi\partial_t\xi - \xi\partial_t\chi\right) = 0,$$

where the bilinear density is $\varrho_{mn} = \chi\partial_t\xi - \xi\partial_t\chi$ and its corresponding flow is $\mathbf{\Phi}_{\varrho mn} = \xi\partial_z\chi - \chi\partial_z\xi$. The subindices stand for the nth and mth order derivative of the original differential equation, $\partial_z^2\psi - v^{-2}\partial_t^2\psi = 0$. Clearly an infinite set of continuity equations of this form can be obtained. An analogous procedure could in fact be also applied to the algorithm leading to the first order derivative expression.

3 Internal consistency

The question then arises as to how to choose the appropriate continuity equation to represent energy conservation. Since the kinetic energy involves a quadratic first-order time derivative and the total energy density must include a kinetic energy term, the possibilities are greatly narrowed. However, the complementary fields equation (6) as well as the continuity equation (10), exhibit such a term. However, their potential energy term is different and this is where the controversy comes in.

The starting point in order to establish a wave equation for the transverse displacement, was to equate the transverse force on a segment with the mass acceleration product. From the wave equation, a conservation equation is obtained that involves both, the kinetic and the potential energy. From these quantities, the acceleration and the force may be obtained. In particular, the force evaluated from minus the gradient of the potential should be equal to the force proposed in the first place in order to obtain the wave equation. The sequence of this theoretical framework is depicted in Fig. 2. This procedure, as we shall see, singles out one of the continuity equations from the infinite set.

3.1 Force derived from ϱ_{pot} potential

The force in a conservative system is given by $\mathbf{F} = -\nabla V$, where V is a scalar potential function. When this function depends explicitly on position and time, the total energy of the system is not conserved [21, p. 5]. The wave equation was established for the transverse string oscillation ψ. The longitudinal displacement of the segments has been neglected in this simple model. The relevance of the longitudinal coupling in more realistic models has been addressed by different authors [1, 9, 22]. Here, the problem is restricted solely to the gradient in the y transverse direction, $F_y = -(\nabla V)_y = -\partial_y V$. The transverse force component derived from the ϱ_{pot} function is

$$F_y\left(\varrho_{\text{pot}}\right) = -(\nabla V)_y = \frac{1}{2}\partial_y\left(T\psi\partial_z^2\psi\right). \tag{13}$$

The force required for a taut string to satisfy a wave equation was obtained from the net force on a string infinitesimal segment (2). These two forces must be the same in order to have a consistent scheme, $F_y(\text{segment}) = F_y\left(\varrho_{\text{pot}}\right)$,

$$T\partial_z^2\psi = \frac{1}{2}\partial_y\left(T\psi\partial_z^2\psi\right). \tag{14}$$

This equation is satisfied if $\partial_z^2\psi = c_1\psi$, where c_1 is constant, since $\frac{1}{2}\partial_y\left(T\psi\partial_z^2\psi\right) = \frac{1}{2}Tc_1\partial_y\left(\psi^2\right) = Tc_1\psi$. Notice that if the derivative of the product on the right hand side is expanded

$$T\partial_z^2\psi = \frac{1}{2}\partial_y\left(T\psi\partial_z^2\psi\right) = \frac{1}{2}\left(T\partial_y\psi\,\partial_z^2\psi\right) + \frac{1}{2}\left(T\psi\partial_y\partial_z^2\psi\right),$$

the term $\frac{1}{2}\left(T\partial_y\psi\,\partial_z^2\psi\right) = \frac{1}{2}\left(T\,\partial_z^2\psi\right)$ cancels out half of the term on the left, since the perturbation ψ for the string is the displacement in the y direction, $\partial_y\psi = 1$. The rightmost term, again provided that $\partial_z^2\psi = c_1\psi$, cancels the remaining part

$$\frac{1}{2}T\partial_z^2\psi = \frac{1}{2}T\psi\partial_y\partial_z^2\psi = \frac{1}{2}T\psi\partial_y\left(c_1\psi\right) = \frac{1}{2}T\psi c_1.$$

For a force satisfying Hooke's linear relationship, $\mu\partial_t^2\psi = -\mu\omega^2\psi$. The string wave equation (3) then becomes,

$$\partial_z^2\psi = -\frac{\mu\,\omega^2}{T}\psi = -k^2\psi. \tag{15}$$

Notice that this result is true for the dynamic condition, i.e., $\partial_t^2\psi \neq 0$. The constant c_1 is then equal to $-k^2$. The transverse force in the y direction is therefore

$$F_y\left(\varrho_{\text{pot}}\right) = -\left(\nabla\varrho_{\text{pot}}\right)_y = \frac{1}{2}T\partial_z^2\psi + \frac{1}{2}T\partial_z^2\psi = T\partial_z^2\psi = -Tk^2\psi.$$

This force, proportional to the second spatial derivative of the perturbation, is identical to (2), thereby producing a

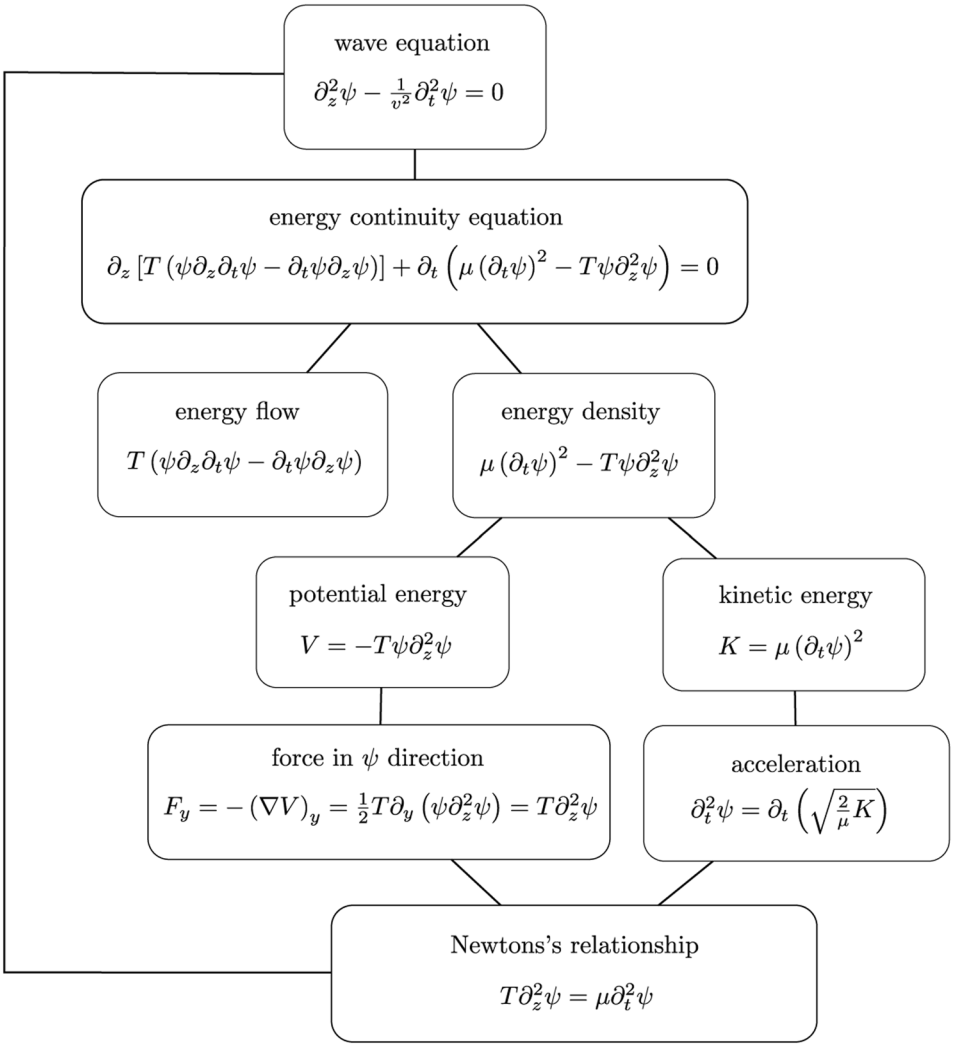

Fig. 2 Internal consistency scheme: a continuity equation is obtained from the infinite set of possibilities. The potential energy derived thereof has to be consistent with the force used in the first place to derive the wave equation

coherent schema. This result makes decisive use of the harmonic condition. An arbitrary well-behaved wave function can be expressed as a sum of harmonic functions via Fourier decomposition. If the oscillating function is not harmonic, the internal consistency scheme can nonetheless be evaluated and fulfilled for each harmonic component. This procedure is illustrated with a two waves interference setup in appendix A.3.

Notice that the present derivation does not involve quasi-static configurations in any way, where the potential energy contribution is separated from the kinetic contribution. Thus, the criticism [16] to the quasi-static approach [4, 23] is not applicable here.

3.2 Force derived from potential with first-order derivative

Evaluate the force from the potential proposed in (11),

$$F_y\left(V_1\right) = -\left(\nabla V_1\right)_y = -\partial_y\left(\frac{1}{2}T\left(\partial_z\psi\right)^2\right).$$

Consider a harmonic wave $\psi = a\cos(kz - \omega t)$, so that $\left(\partial_z\psi\right)^2 = k^2\left(1 - \psi^2\right)$,

$$F_y\left(V_1\right) = -\frac{1}{2}Tk^2\partial_y\left(1 - \psi^2\right) = Tk^2\psi. \tag{16}$$

This force, although linear in ψ as expected, has the opposite sense of the required restitutive force, $F_y = \mu\partial_t^2\psi = -\mu\omega^2\psi$. Therefore, a consistent scheme is not attained with the quadratic potential. Nonetheless, the quantity $\mathcal{E}_1 = \mathcal{E}_1^{\text{kin}} + V_1 = \frac{1}{2}\mu\left(\partial_t\psi\right)^2 + \frac{1}{2}T\left(\partial_z\psi\right)^2$ is a perfectly sound locally conserved quantity. The continuity equation that it satisfies with its corresponding flow $P_1 = -T\partial_z\psi\partial_t\psi$, is certainly correct. What remains to be elucidated, is the physical meaning of these quantities. In the Lagrangian formalism, V_1 is recognized as a generalized velocity dependent potential.

4 Discussion and conclusions

The continuity equation involving a second-order spatial derivative for the potential energy has been derived with the complementary fields formalism. The two terms comprising the flow $\mathbf{\Phi}_\varrho$ in this scheme, have been shown to represent the kinetic and potential energy flows. The internal consistency of the force evaluated from the new derived potential density $\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi$, and the force used to obtain the wave equation has been proved.

The scalar quantities ϱ and $\mathcal{E}_1$, that are the two candidates to represent energy density, satisfy a continuity equation in 1 + 1dimensions, namely (6) and (10), respectively. They are both defined in terms of a perturbation ψ that satisfies the 1 + 1dimensional dispersion-less wave equation. The wave equation for the transverse motion of a string is obtained from Newton's $F_y = \mu\partial_t^2\psi$ equation, where the force F_y is equal to $T\partial_z^2\psi$. This expression, in turn, is obtained from the sum of transverse forces exerted at the two ends of an infinitesimal string segment. The differential form of the conservation equations insures that ϱ and $\mathcal{E}_1$ are locally conserved quantities. Their corresponding flow $\mathbf{\Phi}_1$ or P_1 represents the transport of the assessed density ϱ or $\mathcal{E}_1$, respectively.

The kinetic energy term is identical in ϱ and $\mathcal{E}_1$. However, the remaining energy term, ϱ_{pot} or V_1 differs in the two continuity equations. The working hypothesis has been that this remaining term represents the potential energy of the system. It is well known that the spatial integration of these two "potential energies" differ by a term evaluated at the boundary.

Since the force is defined as minus the gradient of the potential energy $F_y = -\partial_y V$, the crucial test is then whether ϱ_{pot} and/or V_1 satisfy this relationship. We have shown that ϱ_{pot} satisfies the $F_y = T\partial_z^2\psi = -\partial_y\varrho_{\text{pot}}$ equation; therefore, up to an integration constant, ϱ_{pot} represents the potential energy of the system. In contrast, $-\partial_y V_1 \neq F_y$; thus, it cannot represent the potential energy of the system contrary to what has been assumed for so many years.

The possibility of $\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi$ representing the potential energy has been refuted on several grounds. Two of them have been favorably resolved here, namely an adequate continuity equation involving ϱ_{pot} and the dismissal of the quasi-static condition. However, it remains to be seen whether other criticisms, in particular the potential energy distribution in specific cases, are surmountable or not. Some particular configurations are undertaken in the appendix. A notable asset, shown in appendix A.1.1, is that the energy of a harmonic oscillator is obtained when the string's harmonic motion is evaluated at constant z. More refined approximations within the present scheme are certainly desirable. For example, the inclusion of longitudinal motion and possibly dissipation. However, this account has been deliberately kept as simple as possible in order highlight the essential features of the problem.

It has been suggested that the potential energy of only a portion of the string is not uniquely determined [4]. However, for given initial conditions in the sense of the Cauchy problem [24], ψ is uniquely determined by the wave equation and then, the linear partial differential equation $F_y = -\partial_y V(\psi)$ has a unique solution up to integration constants. Therefore, there cannot be any ambiguity in the potential energy distribution within the string. This conclusion has also been recently reached following different arguments by other authors [3, 25].

Regarding other mechanical systems, the potential energy of water waves, in particular ocean waves, is usually evaluated with a quasi-static approach excluding the particle path or its velocity [26]. Although some alternatives have been proposed, the static potential energy calculation still prevails [27, Sec. 4.4.1]. The hydrodynamic equations for deep water gravity waves can be linearized assuming an incompressible nonviscous fluid. The present formalism is then a good candidate to evaluate the potential energy distribution in ocean waves. Although average values for the kinetic and potential energies are often used, the detailed time and space resolved dynamics could have an impact on the design of efficient energy conversion devices.

Compliance with ethical standards

Conflict of interest The corresponding author states that there is no conflict of interest.

A Appendix: Examples of wave profiles

Several arguments that have been put forward in order to choose one potential or the other are physical arguments that are better illustrated for specific wave profiles and boundary conditions.

A.1 Traveling harmonic wave

Consider a simple harmonic traveling wave in the real representation $\psi = a\cos(kz - \omega t)$. The kinetic energy density is

$$\varrho_{\text{kin}} = \frac{1}{2}\mu\left(\partial_t\psi\right)^2 = \frac{1}{2}\mu\, a^2\omega^2\sin^2(kz - \omega t) \tag{17a}$$

and the potential energy density is

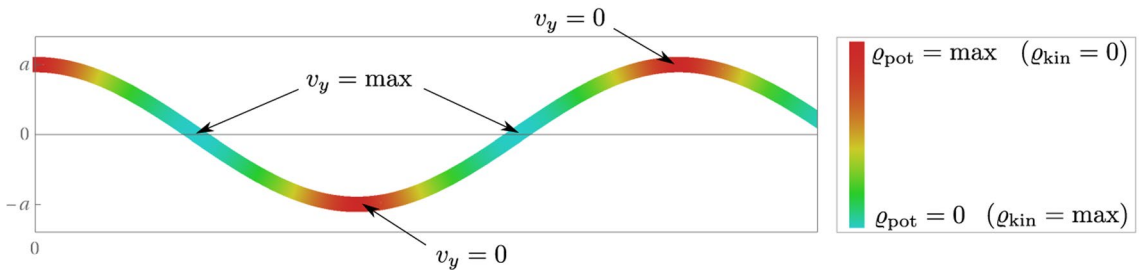

Fig. 3 Traveling harmonic wave. The potential energy $\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi$ is maximum at the crests and troughs, whereas the transverse velocity $v_y = \partial_y\psi$, and hence, the kinetic energy is zero at these points. As the string crosses the equilibrium position (inflection points), the potential energy is zero and the kinetic energy is maximum. Notice the consistency with the forces exerted on the string segments, as depicted in Fig. 1

$$\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi = \frac{1}{2}\mu\, a^2v^2k^2\cos^2(kz-\omega t). \tag{17b}$$

These energies are depicted in a colour density plot as a function of position in Fig. 3. The total energy density is

$$\varrho = \frac{1}{2}\mu\, a^2\left(\omega^2\sin^2(kz-\omega t) + v^2k^2\cos^2(kz-\omega t)\right) = \frac{1}{2}\mu a^2\omega^2. \tag{17c}$$

Kinetic and potential energy are $\frac{\pi}{2}$ out of phase, their sum is constant both in time and space. In particular, at the crests and troughs, where the curvature is maximum, the potential energy is maximum while the kinetic energy is zero. At the inflection points, the potential energy is zero (although the curve slope is finite) while the kinetic energy is maximum. In contrast, in the first-order spatial derivative expression both terms are in phase; this outcome is most peculiar at the curve extrema ($\partial_z\psi = 0$), where kinetic and potential energies are both zero. The energy flow in the present description is

$$\mathbf{\Phi}_\varrho = \frac{1}{2}T\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right) = \frac{1}{2}\mu a^2\omega^2 v. \tag{18}$$

The flow is also time and space independent, even though no averages have been performed. These results are a hallmark of the complementary fields approach, whereby the assessed quantity, in this case energy itself, is flowing to and fro between two forms of energy. In the electromagnetic realm, the helicity or the chirality of the fields have been shown to behave in an analogous fashion [28].

A.1.1 Comparison with harmonic oscillator

The non-propagating problem of a harmonic oscillator should be obtained from the string's harmonic oscillation when evaluated for constant z_i. From the string wave equation (3) and the harmonic condition for the spatial dependence (15),

$$\mu\partial_t^2\psi = T\partial_z^2\psi = -\mu\omega^2\psi \quad \Rightarrow \quad \partial_t^2\psi(t,z_i) + \omega^2\psi(t,z_i) = 0, \tag{19}$$

where the latter is the archetypal harmonic oscillator equation. Physically, each point (segment) in the string executes harmonic motion for an arbitrary fixed z_i. It is well established that the kinetic and potential energies for the harmonic oscillator are $\frac{\pi}{2}$ out of phase. In a mass spring system, the kinetic energy is maximum when the mass passes through the equilibrium position (zero potential energy) and null for maximum displacement when the potential energy in the string is maximum (either compression or elongation). The energy of the system is the same for all times. The string potential energy for a harmonic wave (17b) can be written as $\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi = -\frac{1}{2}T\psi\left(-\frac{\mu\omega^2}{T}\psi\right) = \frac{1}{2}\mu\omega^2\psi^2$. The total energy of the system ϱ, then becomes a sum of squares

$$\varrho = \frac{1}{2}\mu\left(\partial_t\psi\right)^2 + \frac{1}{2}\mu\omega^2\psi^2,$$

that is identical to the kinetic and potential energy terms of the harmonic oscillator. It is then clear that no averaging is required to obtain a time-independent energy since the kinetic and potential contributions are out of phase. The choice of different z_i's merely adds a constant phase shift for the oscillating segment at one z position to another in a different position.

In contrast, notice that for constant z_i, the energy obtained with the first order spatial derivative expression $\mathcal{E}_1 = \frac{1}{2}\mu\left(\partial_t\psi\right)^2 + \frac{1}{2}T\left(\partial_z\psi\right)^2$ does not yield the harmonic oscillator energy but time-dependent kinetic and potential in phase terms. It is only the average of $\langle\mathcal{E}_1\rangle = \frac{1}{2}\mu\omega^2a^2$ and its corresponding average flow $\langle P_1\rangle = \frac{1}{2}\mu\omega^2a^2v$, that reproduce results equivalent to ϱ and $\mathbf{\Phi}_\varrho$ given by (17c) and (18).

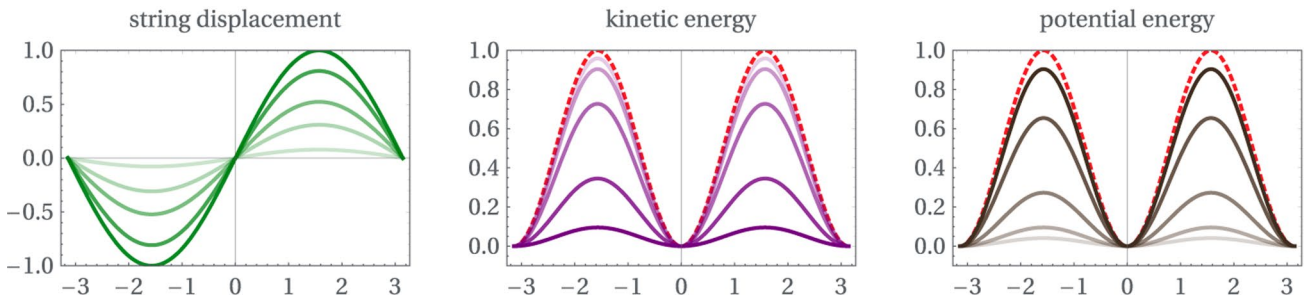

Fig. 4 Counter-propagating harmonic waves with end fixed points at $kz = \pm\pi$. Spatial distribution of the string displacement ψ, kinetic ϱ_{kin} and potential ϱ_{pot} energies at various times (in different shades of colour). When ϱ_{kin} is large (light shade), ϱ_{pot} is small (light shade) and vice versa for dark shading. The total energy ϱ (dotted red) is spatially modulated but constant in time

A.2 Counter-propagating waves with equal amplitude and frequency

It can be misleading, from our point of view, to use the word "standing" for waves propagating in opposite directions because waves are never stationary. Names are sometimes at the inception of misconceptions. If counter-propagating waves have equal amplitudes, the net flow is zero, but there is certainly flow from each of both waves. An altogether different but illustrating analogue common in our crammed cities are intra-city motorways, where two very different scenarios have zero net flow: i) A traffic jam involving all vehicles so that every car is standing and there is zero net flow; in contrast, ii) a fast and steady equal flow of cars in both senses, where the net flow is again zero. Counter-propagating waves are akin to the latter situation where we would hardly speak of anything "standing." It is sometimes mentioned that the name comes from the spatially stationary nodes. An unfair coining, for these are the only stationary points in the 2π interval, while all the rest are wiggling up and down. Leaving aside this digression, consider two counter-propagating but otherwise equal waves represented by

$$\psi(z,t) = a\sin(kz)\sin(\omega t). \tag{20}$$

The kinetic energy density is

$$\varrho_{\text{kin}} = \frac{1}{2}\mu\, a^2\omega^2 \sin^2(kz)\cos^2(\omega t) \tag{21a}$$

and the potential energy density is

$$\varrho_{\text{pot}} = \frac{1}{2}\mu\, a^2 v^2 k^2 \sin^2(kz)\sin^2(\omega t). \tag{21b}$$

The total energy density is

$$\varrho = \frac{1}{2}\mu\, a^2\omega^2 \sin^2(kz). \tag{21c}$$

The total energy density is time independent, goes to zero at $z = 0$ and $kz = \pm\pi$, mod π, as depicted in Fig. 4. The kinetic energy is zero at the nodes, and in particular, at the end points. However, in sharp difference with the first order derivative expression, the potential energy is also zero at the nodes or the end points. The nodes correspond to inflection points were the second spatial derivative vanishes. In particular, there is no curvature at the end points. Every infinitesimal segment in the string executes harmonic transverse motion in the time domain. The amplitude of a given segment increases as it lies further apart from a node. The energy maxima are located at $kz = \pm\frac{\pi}{2}$, mod π. At these points the amplitude is largest. At maximum displacement $kz = \frac{\pi}{2}$, $\omega t = \frac{\pi}{2}$, the kinetic energy is zero but the potential energy is maximum. The energy flow vanishes everywhere

$$\mathbf{\Phi}_{\varrho} = \frac{T}{8}a^2(\sin(2kz)\sin(2\omega t)\omega k - \omega\sin(2kz)k\sin(2\omega t)) = 0. \tag{22}$$

There is no net flow of energy either locally nor globally, even in the sub-wavelength scale. The net flow is time independent and equal to zero for an arbitrary segment of the string. The time independence of the energy density ϱ either for running (17c) or counter-propagating (21c) waves, is in accordance with a conservative force mechanical system, where the sum of kinetic and potential energy should be time independent.

In contrast, the energy density involving the first order spatial derivative for counter-propagating waves is $\mathcal{E}_1 = \frac{1}{2}\mu\, a^2\omega^2\left(\sin^2(kz)\cos^2(\omega t) + \cos^2(kz)\sin^2(\omega t)\right)$ and the power is $P_1 = -\frac{1}{4}\mu\, a^2\omega^2 v\sin(2kz)\sin(2\omega t)$. In the sub-wavelength scale there is energy redistribution and flow[14]. However, the flow is zero at the nodes and antinodes[5]. The integrated flows $\mathbf{\Phi}_{\varrho}$ and P_1 are equal if the end points lie at the nodes. However, if an endpoint say b, is freely supported, the boundary term does not vanish and is equal to $-\frac{1}{2}T\psi(b)\,\partial_z\psi\big|_b = -\frac{1}{4}Ta^2k\sin(2kb)\sin^2(\omega t)$.

A.3 Superposition

Consider the presence of waves with different frequencies. Two or more waves will exhibit the universal phenomenon of interference. In fact, counter-propagating waves interfere, but it is not so evident due to the frequency degeneracy in the co-linear configuration imposed by the single spatial dimension. Superposition of two waves with non-degenerate frequency is evaluated here. The generalization to an arbitrary number of waves is straightforward. Consider two harmonic traveling waves with different frequencies in the real algebra representation $\psi = a_1 \cos\left(k_1 z - \omega_1 t\right) + a_2 \cos\left(k_2 z - \omega_2 t\right)$. The kinetic energy density is

$$\varrho_{\text{kin}} = \frac{1}{2}\mu\left(\partial_t \psi\right)^2 = \frac{1}{2}\mu\left(a_1^2\,\omega_1^2 \sin^2\left(k_1 z - \omega_1 t\right) + a_2^2\,\omega_2^2 \sin^2\left(k_2 z - \omega_2 t\right)\right) + \mu a_1 a_2\,\omega_1\omega_2 \sin\left(k_1 z - \omega_1 t\right)\sin\left(k_2 z - \omega_2 t\right) \tag{23a}$$

and the potential energy density is

$$\varrho_{\text{pot}} = -\frac{1}{2}T\psi\partial_z^2\psi = \frac{1}{2}\mu\left(a_1^2\omega_1^2 \cos^2\left(k_1 z - \omega_1 t\right) + a_2^2\,\omega_2^2 \cos^2\left(k_2 z - \omega_2 t\right)\right) + \frac{1}{2}\mu a_1 a_2\left(\omega_1^2 + \omega_2^2\right)\cos\left(k_1 z - \omega_1 t\right)\cos\left(k_2 z - \omega_2 t\right), \tag{23b}$$

where the dispersionless relationship $v = \frac{\omega_j^2}{k_j^2} = \frac{T}{\mu}$ has been used. Refined models including different velocities for different frequencies are possible. The total energy density is

$$\varrho = \frac{1}{2}\mu a_1^2\,\omega_1^2 + \frac{1}{2}\mu a_2^2\,\omega_2^2 + \mu a_1 a_2\,\omega_1\omega_2 \sin\left(k_1 z - \omega_1 t\right)\sin\left(k_2 z - \omega_2 t\right) + \frac{1}{2}\mu a_1 a_2\left(\omega_1^2 + \omega_2^2\right)\cos\left(k_1 z - \omega_1 t\right)\cos\left(k_2 z - \omega_2 t\right) \tag{23c}$$

The energy flow is

$$\boldsymbol{\Phi}_\varrho = \frac{1}{2}T\left(\psi\partial_z\partial_t\psi - \partial_t\psi\partial_z\psi\right) = \frac{1}{2}\mu\left(a_1^2\,\omega_1^2 + a_2^2\,\omega_2^2\right)v + \frac{1}{2}\mu a_1 a_2\left(\omega_1^2 + \omega_2^2\right)v\cos\left(k_1 z - \omega_1 t\right)\cos\left(k_2 z - \omega_2 t\right) + \mu a_1 a_2\,\omega_1\omega_2 v \sin\left(k_1 z - \omega_1 t\right)\sin\left(k_2 z - \omega_2 t\right). \tag{23d}$$

It is reassuring that $\boldsymbol{\Phi}_\varrho = \varrho\, v$, as expected. The oscillating terms are of course, the manifestation of the waves interference. The energy density will vary in time and space, depending on whether the waves, for a given time at a given position, interfere constructively or destructively. In an experimental realization, dispersion is inevitable with the concomitant wave packet spreading. For several purposes, it is sufficient and convenient to work with average quantities. The temporal average is usually performed. The average energy density is

$$\bar{\varrho} = \frac{1}{2}\mu a_1^2\,\omega_1^2 + \frac{1}{2}\mu a_2^2\,\omega_2^2, \tag{24}$$

Whereas the average flow is

$$\bar{\boldsymbol{\Phi}}_\varrho = \frac{1}{2}\mu\left(a_1^2\,\omega_1^2 + a_2^2\,\omega_2^2\right)v. \tag{25}$$

The non-degenerate frequency functions are orthogonal in the Sturm–Liouville sense, and thus, their contribution will vanish when integrating over the period of the superposed fields. The density and flow coming from the complementary fields continuity equation are independent from the internal consistency scheme. However, it may be of concern that the harmonic condition was imposed in the latter derivation. From the Fourier decomposition, $\psi = \psi_1 + \psi_2$, where $\psi_1 = a_1 \cos\left(k_1 z - \omega_1 t\right)$ and $\psi_2 = a_2 \cos\left(k_2 z - \omega_2 t\right)$. For each component, $F_y\left(\varrho_{\text{pot}}\left(\psi_1\right)\right) = T\partial_z^2\psi_1 = -Tk_1^2\psi_1$ and $F_y\left(\varrho_{\text{pot}}\left(\psi_2\right)\right) = T\partial_z^2\psi_2 = -Tk_2^2\psi_2$. The interference terms deter us from performing this ansatz directly on the superposed solution.

References

1. Narasimha R (1968) Non-linear vibration of an elastic string. J Sound Vib 8(1):134
2. Main IG (1994) Vibrations and waves in physics, 3rd edn. CUP, Cambridge
3. Butikov EI (2012) Misconceptions about the energy of waves in a strained string. Phys Scr 86(3):035403
4. Morse PM, Feshbach H (1953) Methods of theoretical physics. McGraw-Hill, New York
5. Butikov EI (2013) Peculiarities in the energy transfer by waves on strained strings. Phys Scr 88(6):065402
6. Borowitz S (1967) Fundamentals of quantum mechanics. W.A. Benjamin, New York
7. Truesdell C, Toupin RA (1960) The classical field theories, Handbuch der Physik, vol III. Springer, Berlin
8. Burko LM (2010) Energy in one-dimensional linear waves in a string. Eur J Phys 31(5):L71
9. Rowland DR (2013) Small amplitude transverse waves on taut strings: exploring the significant effects of longitudinal motion on wave energy location and propagation. Eur J Phys 34(2):225
10. Fernández-Guasti M (2004) Complementary fields conservation equation derived from the scalar wave equation. J Phys A Math Gen 37:4107–4121
11. Fernández-Guasti M, Gil-Villegas A (2002) Orthogonal functions invariant for the time dependent harmonic oscillator. Phys Lett A 292:243–245
12. Fernández-Guasti M (2018) Energy content in time dependent linear restoring force systems. Phys Lett A 382(45):3231–3237
13. Fernández-Guasti M (2019) Gauge invariance of the helicity continuity equation. Ann Phys 406:186–199
14. Repetto CE, Roatta A, Welti RJ (2011) Energy in one-dimensional linear waves. Eur J Phys 32(6):L39
15. Mathews WN (1985) Energy in a one-dimensional small amplitude mechanical wave. Am J Phys 53(10):974
16. Butikov EI (2011) Reply to comments on 'energy in one-dimensional linear waves in a string'. Eur J Phys 32(6):L35–L45
17. Kibble TWB (1965) Conservation laws for free fields. J Math Phys 6(7):1022

18. Fairlie DB (1965) Conservation laws and invariance principles. Il Nuovo Cimento (1955–1965) 37(3):897
19. Lipkin DM (1964) J Math Phys 5(5):696
20. Ragusa S (1988) New first-order conservation laws for the electromagnetic field. Il Nuovo Cimento B (1971–1996) 101(1):121
21. Goldstein H, Poole C, Safko J (2002) Classical mechanics, 3rd edn. Addison Weseley, Boston
22. Rowland DR (2011) The potential energy density in transverse string waves depends critically on longitudinal motion. Eur J Phys 32(6):1475
23. Burko LM (2011) Reply to comments on 'energy in one-dimensional linear waves in a string'. Eur J Phys 32(6):L43
24. Salsa S (2009) Partial differential equations in action. Springer, Berlin
25. Rowland DR (2015) Comment on: 'peculiarities in the energy transfer by waves on strained strings'. Phys Scr 90(8):087002
26. Calisal S (1983) A note on the derivation of potential energy for two-dimensional water waves. Ocean Eng 10(2):133
27. Falnes J (2004) Ocean waves and oscillating systems. CUP, Cambridge
28. Fernández-Guasti M (2019) Chirality, helicity and the rotational content of electromagnetic fields. Phys Lett A 383:3180–3186

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