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# A Hyperbolic Non Distributive Algebra in $1+2$ Dimensions

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**Abstract.** We introduce a non distributive algebra over the reals in 1 + 2 dimensions that contains the hyperbolic complex algebra $\mathbb{H}_2$. The algebra has divisors of zero that can be avoided by introducing the necessary conditions. Under these conditions, the proposed addition and product operations satisfy group properties. More stringent restrictions sufficient to satisfy group properties separate the algebra in two subspaces. As an application, the composition of velocities in a deformed Lorentz metric is presented. In this approach, Minkowski light cones are deformed into light bipyramids.

**Keywords.** Algebras, non-distributive algebras, hypercomplex numbers.

## 1. Introduction

Systems of hypercomplex numbers, such as quaternions or octonions are algebras over the real numbers that lack some properties for their product, either commutativity or associativity. The search for higher dimensional algebras retaining some of these properties is severely limited by Frobenius and Hurwitz theorems. The former theorem establishes that finite-dimensional associative division algebras over real numbers must be isomorphic to the real $\mathbb{R}$, complex $\mathbb{C}$ or quaternion $\mathbb{H}$ algebras with dimensions 1, 2, and 4, respectively. The latter theorem states that the only normed division algebras over $\mathbb{R}$ are the real $\mathbb{R}$, complex $\mathbb{C}$, quaternion $\mathbb{H}$ and the octonions $\mathbb{O}$. Of these algebras quaternions are no longer commutative and octonions are neither commutative nor associative.

These theorems require the absence of zero divisors. If distributivity and associativity are fulfilled then the algebra (except for reals and complex) is either not commutative or has divisors of zero. A theorem due to Scheffers establishes that for distributive systems with unity, the differential calculus

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does exist only if the systems are commutative [4]. It is still an open question whether non-distributive, associative and commutative algebras support some sort of differential calculus. Differential calculus is crucial in order to define functions of a hypercomplex variable and to associate them with the infinite-dimensional Lie group of functional mappings. For this reason there has been renewed interest in commutative algebras such as Segre's commutative quaternions although they possess zero divisors [5].

Two dimensional hypercomplex numbers contain a real component and an imaginary component. Depending on the product definition of imaginary unit, these hypercomplex numbers are elliptic, parabolic or hyperbolic. Elliptic hypercomplex numbers are the complex numbers, a commutative division algebra. Hyperbolic hypercomplex two dimensional numbers or split-complex numbers form an associative and commutative algebra that is tailor made to describe Minkowski space-time special relativity in $1+1$ dimensions. Higher dimensional hyperbolic numbers have been explored using products with zero divisors as a basis [10].

In this paper, we introduce an algebra in $1+2$ dimensions that contains the algebra of hyperbolic $\mathbb{H}_2$ numbers as we shall see in Lemma 5.1. We call this algebra a *hyperbolic scator* or *real scator* algebra, and we will see that its product is commutative and associative. However, as we shall see, the product does not distribute over addition but in some special cases. Moreover, the product has zero divisors and it is defined only on a proper subset of $\mathbb{R}^3$. The construction can be easily generalized to any dimension, but many of its properties are straightforward generalizations of the 3-dimensional case.

## 2. Three-Dimensional Scator Algebra

Consider the set of all expressions of the form

$$\varphi = (f_0; f_1, f_2) \in \mathbb{R}^{1+2} \qquad \text{with } f_i \in \mathbb{R},$$

with addition defined componentwise. The product of two elements $\alpha = (a_0; a_1, a_2), \beta = (b_0; b_1, b_2)$ in $\mathbb{R}^{1+2}$, say $\alpha \cdot \beta = \gamma = (g_0; g_1, g_2) \in \mathbb{R}^{1+2}$ is given by

$$g_0 = a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1b_1a_2b_2}{a_0b_0} = a_0b_0\left(1 + \frac{a_1b_1}{a_0b_0}\right)\left(1 + \frac{a_2b_2}{a_0b_0}\right) \quad (2.1a)$$

$$g_1 = \left(1 + \frac{a_2b_2}{a_0b_0}\right)(a_0b_1 + a_1b_0) \quad (2.1b)$$

$$g_2 = \left(1 + \frac{a_1b_1}{a_0b_0}\right)(a_0b_2 + a_2b_0). \quad (2.1c)$$

In this formulation, the condition for the definition of a product is that

$$a_0b_0 \neq 0 \quad \text{if} \quad a_1b_1 \neq 0 \text{ and } a_2b_2 \neq 0. \quad (2.2)$$

We call the elements $\varphi = (f_0; f_1, f_2)$ *scators* and we will show that, with some restrictions they form a weak hypercomplex system, since the product is not distributive.

*Remarks* 2.1. To begin with, it is immediate that:

(1): We can identify the elements of the field $\mathbb{R}$ with the scators of the form $\alpha = (a_0; 0, 0)$, since addition is componentwise and multiplication of two of these elements, say $\alpha$ and $\beta = (b_0; 0, 0)$, assuming that $a_0 b_0 \neq 0$, is given by

$$\alpha \cdot \beta = (a_0; 0, 0) \cdot (b_0; 0, 0) = (a_0 b_0; (1 + 0)(0), (1 + 0)(0)) = (a_0 b_0; 0, 0).$$

Sometimes we will use the identification $(a_0; 0, 0) = a_0$ and call the scator $(a_0; 0, 0)$ a *scalar*. In general, given a scator $\alpha = (a_0; a_1, a_2)$ we call $a_0$ the scalar component, and $a_1, a_2$ the director components of the scator $\alpha$.

(2): If $\alpha = (a_0; 0, 0)$ is identified with $a_0 \in \mathbb{R} - \{0\}$ and if $\beta = (b_0; b_1, b_2)$ is any scator (with $a_0 b_0 \neq 0$), we have

$$\begin{aligned} \alpha \cdot \beta &= (a_0; 0, 0) \cdot (b_0; b_1, b_2) \\ &= (a_0 b_0; (1)(a_0 b_1 + 0 \cdot b_0), (1)(a_0 b_2 + 0 b_0)) \\ &= (a_0 b_0; a_0 b_1, a_0 b_2). \end{aligned}$$

In other words, the product of a scalar with a scator is componentwise.

(3): In particular, for the scalar $(1; 0, 0) = 1 \in \mathbb{R}$ and for any scator $\varphi = (f_0; f_1, f_2)$ we have that 1 is neutral, i.e., $1 \cdot \varphi = \varphi$.

(4) The product of a scalar with the sum of two scators is distributive.

(5): The product of the sum of two scalars with a scator is distributive.

(6): Whenever defined, the product of two scators is commutative. This follows from the symmetry of the product definition.

These remarks mean that the set of scators behaves as an algebra over the real field. However, we have the restriction (2.2) for the product and, as we shall see, in general the scator product is not distributive, and there are zero divisors.

In the definition of the product, the $a_0 b_0$ terms in the denominators give rise to undefined quantities if not treated carefully, i.e., imposing condition (2.2). Furthermore, divisors of zero are related to these terms as we shall presently see. It is therefore tempting to propose a product definition without these denominators. However, if $g_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_1 b_1 a_2 b_2$, and similar expressions without $a_0, b_0$ are defined for the other components, the product is no longer associative, as can be seen by direct computation.

**Conjugate and norm.** The *conjugate* of $\varphi = (f_0; f_1, f_2)$ is

$$\varphi^* = (f_0; -f_1, -f_2)$$

and it is immediate that $(\varphi^*)^* = \varphi$, $(\alpha + \beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \alpha^* \beta^*$. Moreover, $\varphi \in \mathbb{R}$ if and only if $\varphi^* = \varphi$.

The *norm* or *modulus* squared of $\varphi = (f_0; f_1, f_2)$ is

$$\|\varphi\|^2 = \varphi \cdot \varphi^* = a_0^2 \left(1 - \frac{a_1^2}{a_0^2}\right) \left(1 - \frac{a_2^2}{a_0^2}\right). \tag{2.3}$$

Thus, the squared norm of a scator is a scalar, $\|\varphi\| = \|\varphi^*\|$ and if $\varphi = (a_0; a_1, a_2)$ we have that

$$a_0 = \frac{1}{2}\left(\varphi + \varphi^*\right).$$

Moreover, direct computation shows that the norm of a product is the product of the norms, provided that zero divisors are excluded.

**Inverses.** Given two scators $\alpha = (a_0; a_1, a_2), \beta = (b_0; b_1, b_2)$ with $a_0 b_0 \neq 0$, for their product to be 1, we need that the first component $g_0 = 1$ and the other components $g_1 = g_2 = 0$:

$$g_0 = a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right) = 1 \tag{2.4a}$$

and the director components of the product are

$$g_1 = \left(1 + \frac{a_2 b_2}{a_0 b_0}\right)(b_0 a_1 + a_0 b_1) = 0, \tag{2.4b}$$

$$g_2 = \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)(b_0 a_2 + a_0 b_2) = 0. \tag{2.4c}$$

Observe now that if the components of the first factor in (2.4b) and (2.4c) are zero, i.e., if $\left(1 + \frac{a_j b_j}{a_0 b_0}\right) = 0$, then the scalar component is also zero, and thus this possibility is excluded. It follows that the second factors must be zero:

$$(b_0 a_1 + a_0 b_1) = 0 \Rightarrow b_1 = -\frac{b_0 a_1}{a_0}$$

and

$$(b_0 a_2 + a_0 b_2) = 0 \Rightarrow b_2 = -\frac{b_0 a_2}{a_0}.$$

The first component becomes

$$g_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 + \frac{a_1 b_1 a_2 b_2}{a_0 b_0} = a_0 b_0 \left(1 - \left(\frac{a_1}{a_0}\right)^2\right)\left(1 - \left(\frac{a_2}{a_0}\right)^2\right) = 1,$$

from where it follows that

$$b_0 = \frac{1}{a_0 \left(1 - \left(\frac{a_1}{a_0}\right)^2\right)\left(1 - \left(\frac{a_2}{a_0}\right)^2\right)}$$

and hence

$$b_1 = -\frac{b_0 a_1}{a_0} = -\frac{a_1}{a_0^2 \left(1 - \left(\frac{a_1}{a_0}\right)^2\right)\left(1 - \left(\frac{a_2}{a_0}\right)^2\right)}$$

and

$$b_2 = -\frac{b_0 a_2}{a_0} = -\frac{a_2}{a_0^2 \left(1 - \left(\frac{a_1}{a_0}\right)^2\right)\left(1 - \left(\frac{a_2}{a_0}\right)^2\right)}.$$

Now observe that we must exclude $a_0 = 0$, $\frac{a_1}{a_0} = \pm 1$, $\frac{a_2}{a_0} = \pm 1$. We have thus shown that $\beta = (b_0; b_1, b_2)$ is the inverse of $\alpha = (a_0; a_1, a_2)$. This can be rewritten as follows. Consider the conjugate $\alpha^* = (a_0; -a_1, -a_2)$ of $\alpha$. Then, the product

$$\alpha\alpha^* = \left(a_0^2 - a_1^2 - a_2^2 + \frac{a_1^2 a_2^2}{a_0^2}; 0, 0\right)$$

and so the inverse of $\alpha$ is

$$\alpha^{-1} = \frac{1}{a_0^2\left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right)}\alpha^*$$

provided that $\left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right) \neq 0$. Summarizing, the scator $\alpha = (a_0; a_1, a_2)$ has an inverse provided that

$$a_0 \neq \pm a_1, a_0 \neq \pm a_2, \quad a_0 \neq 0 \quad \text{when} \quad a_1 a_2 \neq 0. \tag{2.5}$$

It follows that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

**Zero divisors.** Allow for two non-zero scators $\alpha = (a_0; a_1, a_2)$ and $\beta = (b_0; b_1, b_2)$ such that their product is $\alpha\beta = \gamma = (g_0; g_1, g_2) = (0; 0, 0)$. From the definition of the scator product

$$g_0 = a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right) = 0, \tag{2.6a}$$

where the director components of the product are

$$g_1 = \left(1 + \frac{a_2 b_2}{a_0 b_0}\right)(b_0 a_1 + a_0 b_1) = 0, \tag{2.6b}$$

$$g_2 = \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)(b_0 a_2 + a_0 b_2) = 0. \tag{2.6c}$$

We want to characterize these zero divisors. The cases are:

Case 1. If $a_0 b_0 \neq 0$, we consider the last two factors in (2.6a). We have two subcases: The first subcase 1.1 is when both factors are zero, that is, when the following two conditions are satisfied

$$a_1 b_1 = -a_0 b_0 \tag{2.7a}$$

$$a_2 b_2 = -a_0 b_0, \tag{2.7b}$$

and in this case the three components $g_0, g_1, g_2$ are zero. Thus, two factors that satisfy

$$a_1 b_1 = a_2 b_2 = -a_0 b_0 \tag{2.8}$$

are zero divisors. The second subcase 1.2 is when only one factor in (2.6a) is zero, say $\left(1 + \frac{a_1 b_1}{a_0 b_0}\right) = 0$. Then, $g_0 = g_2 = 0$. In order to have $g_1 = 0$, the second factor in (2.6b) should be 0, that is, $b_0 a_1 + a_0 b_1 = 0$. Multiplying by $a_0$ and using $\left(1 + \frac{a_1 b_1}{a_0 b_0}\right) = 0$ we obtain $a_0 = \pm a_1$ and $b_0 = \mp b_1$. Analogously, when the last factor in (2.6a) is zero, we obtain $a_0 = \pm a_2$ and $b_0 = \mp b_2$. Therefore, there are also zero divisors when either of the following conditions are satisfied

$$a_0 = \pm a_1 \quad \text{and} \quad \frac{a_1}{a_0} = -\frac{b_0}{b_1} \quad \text{for all} \quad a_2, b_2, \tag{2.9a}$$

$$a_0 = \pm a_2 \quad \text{and} \quad \frac{a_2}{a_0} = -\frac{b_0}{b_2} \quad \text{for all} \quad a_1, b_1. \tag{2.9b}$$

From (2.5), it follows that the first conditions in the above equations involve non invertible elements.

Case 2. If $a_0 b_0 = 0$, then either $a_0 = 0$ or $b_0 = 0$. However, as we shall see, each case implies the other one. Indeed, when $a_0 = 0$, in a non-trivial zero divisor $\alpha = (0; a_1, a_2)$. For the product to be well-defined, only one of its director components is non-zero, say $a_0 = 0$, $a_1 \neq 0$, $a_2 = 0$. In this case, from (2.6c), the component $g_2 = 0$, since its second factor is 0. From (2.6b) it follows that $g_1 = b_0 a_1 = 0$ and thus $b_0 = 0$. Therefore, there is only one case $a_0 = 0 = b_0$. Moreover, according to the above argument, $\alpha = (0; a_1, 0)$ or $\alpha = (0; 0, a_2)$ and $\beta = (0; b_1, 0)$ or $\beta = (0; 0, b_2)$. Only the products $(0; a_1, 0)(0; 0, b_2)$ and $(0; 0, a_2)(0; b_1, 0)$ give zero divisors. These two cases are included in equations (2.7a), (2.7b) of case 1.1, because if $a_0 = b_0 = 0$ then $a_1 b_1 = 0$ and $a_2 b_2 = 0$. These last two equations are fulfilled if $a_1 = b_2 = 0$ or $a_2 = b_1 = 0$. This exhausts all cases for the characterization of zero divisors.

Observe that to avoid zero divisors, only invertible elements that satisfy the two conditions

$$a_1 b_1 \neq -a_0 b_0 \tag{2.10a}$$

$$a_2 b_2 \neq -a_0 b_0. \tag{2.10b}$$

should be considered.

## 3. Associativity

**Lemma 3.1.** *The product is associative, when divisors of zero are excluded.*

*Proof.* For $\alpha, \beta, \gamma$ as before, and $\varphi = (f_0; f_1, f_2)$, evaluate $\zeta = (h_0; h_1, h_2) = \gamma\varphi = (\alpha\beta)\,\varphi$. The scalar component from (2.1a) is

$$h_0 = [(\alpha\beta)\,\varphi]_0 = g_0 f_0 \left(1 + \frac{g_1 f_1}{g_0 f_0}\right)\left(1 + \frac{g_2 f_2}{g_0 f_0}\right),$$

and in turn, evaluation of $\gamma$ in terms of the product $\alpha\beta$ gives

$$h_0 = [(\alpha\beta)\,\varphi]_0 = a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right) f_0$$
$$\times \left(1 + \frac{\left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right) f_1}{\left(1 + \frac{a_1 b_1}{a_0 b_0}\right) f_0}\right)\left(1 + \frac{\left(\frac{a_2}{a_0} + \frac{b_2}{b_0}\right) f_2}{\left(1 + \frac{a_2 b_2}{a_0 b_0}\right) f_0}\right).$$

This expression may be rewritten in the nicely symmetrical form

$$\begin{aligned} h_0 &= [(\alpha\beta)\,\varphi]_0 \\ &= a_0 b_0 f_0 \left(1 + \frac{a_1 b_1}{a_0 b_0} + \frac{a_1}{a_0}\frac{f_1}{f_0} + \frac{b_1}{b_0}\frac{f_1}{f_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0} + \frac{a_2}{a_0}\frac{f_2}{f_0} + \frac{b_2}{b_0}\frac{f_2}{f_0}\right), \end{aligned}$$

provided that the factors that are being canceled out in numerator and denominator are not zero, namely

$$\frac{a_1 b_1}{a_0 b_0} \neq -1, \quad \frac{a_2 b_2}{a_0 b_0} \neq -1. \tag{3.1}$$

On the other hand, evaluate $\zeta' = (h_0'; h_1', h_2') = \alpha\,(\beta\varphi)$ and to ease the notation, set $\tau = \beta\varphi = (t_0; t_1, t_2)$. The scalar component from (2.1a) is

$$h_0' = [\alpha\,(\beta\varphi)]_0 = a_0 t_0 \left(1 + \frac{a_1 t_1}{a_0 t_0}\right)\left(1 + \frac{a_2 t_2}{a_0 t_0}\right),$$

and in turn, evaluation of $\tau$ in terms of the product $\beta\varphi$ gives

$$\begin{aligned} h_0' = [\alpha\,(\beta\varphi)]_0 = a_0 b_0 f_0 &\left(1 + \frac{b_1 f_1}{b_0 f_0}\right)\left(1 + \frac{b_2 f_2}{b_0 f_0}\right) \\ &\times \left(1 + \frac{\left(\frac{b_1}{b_0} + \frac{f_1}{f_0}\right) a_1}{\left(1 + \frac{b_1 f_1}{b_0 f_0}\right) a_0}\right)\left(1 + \frac{\left(\frac{b_2}{b_0} + \frac{f_2}{f_0}\right) a_1}{\left(1 + \frac{b_2 f_2}{b_0 f_0}\right) a_0}\right). \end{aligned}$$

This expression may be rewritten in the symmetrical form

$$\begin{aligned} h_0' &= [(\alpha\beta)\,\varphi]_0 \\ &= a_0 b_0 f_0 \left(1 + \frac{a_1 b_1}{a_0 b_0} + \frac{a_1}{a_0}\frac{f_1}{f_0} + \frac{b_1}{b_0}\frac{f_1}{f_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0} + \frac{a_2}{a_0}\frac{f_2}{f_0} + \frac{b_2}{b_0}\frac{f_2}{f_0}\right), \end{aligned}$$

provided that the terms $\frac{b_1 f_1}{b_0 f_0}$ and $\frac{b_2 f_2}{b_0 f_0}$ are different from minus one,

$$\frac{b_1 f_1}{b_0 f_0} \neq -1, \quad \frac{b_2 f_2}{b_0 f_0} \neq -1. \tag{3.2}$$

Comparison of the expressions for $h_0'$ and $h_0$, shows that they are identical, hence the scalar component of the product is associative $[(\alpha\beta)\,\varphi]_0 = [\alpha\,(\beta\varphi)]_0$. Consider now the first director component from (2.1b)

$$h_1 = [(\alpha\beta)\,\varphi]_1 = [(\alpha\beta)\,\varphi]_0 \frac{\left(\frac{g_1}{g_0} + \frac{f_1}{f_0}\right)}{\left(1 + \frac{g_1 f_1}{g_0 f_0}\right)},$$

evaluation of $\gamma$ in terms of $\alpha\beta$ components gives

$$\frac{h_1}{h_0} = \frac{[(\alpha\beta)\,\varphi]_1}{[(\alpha\beta)\,\varphi]_0} = \frac{\left(\frac{\left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)}{\left(1 + \frac{a_1 b_1}{a_0 b_0}\right)} + \frac{f_1}{f_0}\right)}{\left(1 + \frac{\left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)}{\left(1 + \frac{a_1 b_1}{a_0 b_0}\right)}\frac{f_1}{f_0}\right)}.$$

This expression may also be put in a symmetrical fashion

$$\frac{h_1}{h_0} = \frac{[(\alpha\beta)\,\varphi]_1}{[(\alpha\beta)\,\varphi]_0} = \frac{\frac{a_1}{a_0} + \frac{b_1}{b_0} + \frac{f_1}{f_0} + \frac{a_1 b_1 f_1}{a_0 b_0 f_0}}{1 + \frac{a_1 b_1}{a_0 b_0} + \frac{a_1 f_1}{a_0 f_0} + \frac{b_1 f_1}{b_0 f_0}},$$

provided that equations (3.1) are fulfilled. On the other hand, evaluate

$$h_1' = [\alpha\,(\beta\varphi)]_1 = [\alpha\,(\beta\varphi)]_0 \frac{\left(\frac{a_1}{a_0} + \frac{t_1}{t_0}\right)}{\left(1 + \frac{a_1 t_1}{a_0 t_0}\right)},$$

expressing $\tau$ in terms of $\beta\varphi$ components gives

$$\frac{h_1'}{h_0'} = \frac{[\alpha\,(\beta\varphi)]_1}{[\alpha\,(\beta\varphi)]_0} = \frac{\left(\frac{a_1}{a_0} + \frac{\left(\frac{b_1}{b_0}+\frac{f_1}{f_0}\right)}{\left(1+\frac{b_1 f_1}{b_0 f_0}\right)}\right)}{\left(1 + \frac{a_1}{a_0}\frac{\left(\frac{b_1}{b_0}+\frac{f_1}{f_0}\right)}{\left(1+\frac{b_1 f_1}{b_0 f_0}\right)}\right)}.$$

This expression may also be put in a symmetrical fashion

$$\frac{h_1'}{h_0'} = \frac{[\alpha\,(\beta\varphi)]_1}{[\alpha\,(\beta\varphi)]_0} = \frac{\frac{a_1}{a_0} + \frac{b_1}{b_0} + \frac{f_1}{f_0} + \frac{a_1 b_1 f_1}{a_0 b_0 f_0}}{1 + \frac{a_1 b_1}{a_0 b_0} + \frac{a_1 f_1}{a_0 f_0} + \frac{b_1 f_1}{b_0 f_0}},$$

provided that equations (3.2) are fulfilled. Since $h_0' = h_0$, the first component is also associative $h_1' = h_1$, $[(\alpha\beta)\,\varphi]_1 = [\alpha\,(\beta\varphi)]_1$. An analogous procedure leads to the second component associativity $[(\alpha\beta)\,\varphi]_2 = [\alpha\,(\beta\varphi)]_2$. Since all components are associative, the product is then associative

$$(\alpha\beta)\,\varphi = \alpha\,(\beta\varphi)\,,$$

provided that conditions 3.1 and 3.2 are satisfied. □

*Remark* 3.2. The restrictions $\frac{a_1 b_1}{a_0 b_0} \neq -1$, $\frac{a_2 b_2}{a_0 b_0} \neq -1$, $\frac{b_1 f_1}{b_0 f_0} \neq -1$, $\frac{b_2 f_2}{b_0 f_0} \neq -1$ required for associativity to hold are in fact, the conditions required to avoid divisors of zero.

## 4. Restricted Space

The necessary conditions for scators to produce a group under multiplication were established in the two previous sections. There are two interesting, albeit more restrictive conditions, that also achieve this goal.

**Lemma 4.1.** *The restricted space conditions $a_0, b_0 \neq 0$, $a_0^2 > a_1^2, a_2^2$ and $b_0^2 > b_1^2, b_2^2$ are sufficient conditions for the real scator subset in $\mathbb{R}^{1+2}$, to satisfy group properties under the product operation.*

*Proof.* The stronger condition $a_0, b_0 \neq 0$, encompasses the condition required to have a well defined product (2.2). Unit and inverse existence are assured since the inequality of scalar components larger than director components prevents the equalities $a_0^2 = a_1^2$, $a_0^2 = a_2^2$. Divisors of zero are avoided because the restricted space conditions impose $\left(\frac{a_1}{a_0}\right)^2 < 1$ and $\left(\frac{b_0}{b_1}\right)^2 > 1$, hence

$\frac{a_1}{a_0} \neq \pm\frac{b_0}{b_1}$ and similarly for the second components. Therefore conditions (3.1) and (3.2) are fulfilled and associativity holds. To prove closure under the restricted space domain, recall that $1 > \frac{a_j}{a_0}$, thus $1 - \frac{a_j}{a_0} > 0$, similarly $1 - \frac{b_j}{b_0} > 0$. The product of these two terms gives the inequality

$$\left(1 - \frac{a_j}{a_0}\right)\left(1 - \frac{b_j}{b_0}\right) > 0,\ j = 1, 2.$$

The expansion of this product is $1 - \frac{a_j}{a_0} - \frac{b_j}{b_0} + \frac{a_j}{a_0}\frac{b_j}{b_0} > 0$, so that $1 + \frac{a_j}{a_0}\frac{b_j}{b_0} > \frac{a_j}{a_0} + \frac{b_j}{b_0}$. It is possible to divide by $1 + \frac{a_j}{a_0}\frac{b_j}{b_0}$ since this term is not zero,

$$1 > \frac{\frac{a_j}{a_0} + \frac{b_j}{b_0}}{1 + \frac{a_j}{a_0}\frac{b_j}{b_0}}.$$

Comparison with equations (2.1a), (2.1b) and (2.1c) shows that the RHS of the inequality is the quotient $\frac{g_j}{g_0}$ of the product of two scators, thus $g_0 > g_j$. The domain and the codomain of the restricted space are then equal. □

*Remark* 4.2. There is a different restricted space condition, when the director components terms are larger than the scalar term $a_0, b_0 \neq 0$, $a_0^2 < a_1^2, a_2^2$ and $b_0^2 < b_1^2, b_2^2$, where the scator product also satisfies group properties.

## 5. Scator in Terms of a Basis

The scators $1, \hat{\mathbf{e}}_1 = (0; 1, 0), \hat{\mathbf{e}}_2 = (0; 0, 1)$ form a basis for $\mathbb{R}^{1+2}$ and so any scator may be written as

$$\alpha = (a_0; a_1, a_2) = a_0 + a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2.$$

The first component of the first element is necessarily a scalar as we have already proved in item (1) of remark 2.1. The product of two scators $\alpha\beta$ according to the definition (2.1a)-(2.1c) is

$$\alpha\beta = \left(a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1b_1a_2b_2}{a_0b_0}\right) + \left(1 + \frac{a_2b_2}{a_0b_0}\right)(b_0a_1 + a_0b_1)\,\hat{\mathbf{e}}_1 + \left(1 + \frac{a_1b_1}{a_0b_0}\right)(b_0a_2 + a_0b_2)\,\hat{\mathbf{e}}_2. \quad (5.1)$$

This operation is not bilinear and therefore cannot be written in terms of the product of its addends. To wit, if the product were distributed over the scator components the following nine terms would be obtained

$$\alpha\beta = a_0b_0 + a_0b_1\hat{\mathbf{e}}_1 + a_0b_2\hat{\mathbf{e}}_2 + b_0a_1\hat{\mathbf{e}}_1 + b_0a_2\hat{\mathbf{e}}_2 + b_1a_1\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + b_2a_1\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + b_1a_2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + b_2a_2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2.$$

However, each of the director products $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_1\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2$ cannot be rewritten in the general form $x + y\hat{\mathbf{e}}_1 + z\hat{\mathbf{e}}_2$ with $x, y, z$ independent of the scator coefficients in order to reproduce the required product terms in (5.1).

**Lemma 5.1.** *Hyperbolic* $1+1$ *dimensional scators, that is with only one non-vanishing director component either* $\hat{\mathbf{e}}_1$ *or* $\hat{\mathbf{e}}_2$*, are identical to double numbers, as defined in* [11].

*Proof.* Consider the case of the product of two scators with only one and the same non zero director component, say $\hat{\mathbf{e}}_2$ (but could equally be $\hat{\mathbf{e}}_1$), from the scator product definition (2.1a)-(2.1c)

$$\alpha\beta = (a_0b_0 + a_2b_2; 0, b_0a_2 + a_0b_2) = a_0b_0 + a_2b_2 + (b_0a_2 + a_0b_2)\,\hat{\mathbf{e}}_2.$$

The product thus becomes identical to the product of hyperbolic complex or double numbers. The product can then be distributed over the scator components

$$\alpha\beta = (a_0 + a_2\hat{\mathbf{e}}_2)\,(b_0 + b_2\hat{\mathbf{e}}_2) = a_0b_0 + a_2b_2 + (b_0a_2 + a_0b_2)\,\hat{\mathbf{e}}_2$$

provided that $\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 = 1$. The addition operation is the same component-wise for scators and double numbers. 1+1 hyperbolic scators with either $\hat{\mathbf{e}}_1$ or $\hat{\mathbf{e}}_2$ thus form a commutative ring identical to double $\mathbb{H}_2$ numbers. □

*Remark* 5.2. The equality $\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 = 1$ is consistent with the scator product definition (2.1a), since the product $(0; 0, 1)\,(0; 0, 1)$ is equal to $(1; 0, 0)$. Care should be taken to make the null director component zero before the scalar component is taken to the zero limit.

On the other hand, if we consider the particular case of the product of two scators with different non-vanishing director component from the scator product definition

$$\alpha\beta = (a_0; a_1, 0)\,(b_0; 0, b_2) = (a_0b_0; b_0a_1, a_0b_2)\,.$$

In terms of the basis $1, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$, this operation is

$$\alpha\beta = (a_0 + a_1\hat{\mathbf{e}}_1)\,(b_0 + b_2\hat{\mathbf{e}}_2) = a_0b_0 + a_1b_0\hat{\mathbf{e}}_1 + a_0b_2\hat{\mathbf{e}}_2$$

The product can then be distributed over the scator components provided that $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 = 0$. This last equality is consistent with the scator product definition, namely the product $(0; 1, 0)\,(0; 0, 1)$ is equal to $(0; 0, 0)$. Then we can construct table 1 with the product of scators with a single non-vanishing component. Notice that this table does not contain all the information needed to define the product of two arbitrary scators with two non-vanishing components.

Recall that the characteristic matrix of a hypercomplex algebra is generated with the bi-product of all the components in the director basis. The multiplication table 1 describes the product of 1+2 hyperbolic scators with a single non-vanishing component. This is as close as we can get to a characteristic matrix. However, it is of limited use because the product of two 1+2 hyperbolic scators cannot be obtained from this matrix as we shall presently show due to the lack of distributivity.

**Lemma 5.3.** *The product of two arbitrary* $1+2$ *hyperbolic scators cannot be represented by a matrix-matrix product.*

| | 1 | $\hat{\mathbf{e}}_1$ | $\hat{\mathbf{e}}_2$ |
|---|---|---|---|
| 1 | 1 | $\hat{\mathbf{e}}_1$ | $\hat{\mathbf{e}}_2$ |
| $\hat{\mathbf{e}}_1$ | $\hat{\mathbf{e}}_1$ | 1 | 0 |
| $\hat{\mathbf{e}}_2$ | $\hat{\mathbf{e}}_2$ | 0 | 1 |

Table 1. Product of single director elements. This table should be read with care since it does not contain all the information needed to define the product of two arbitrary scators.

*Proof.* Suppose that the product of two $1+2$ hyperbolic scators can be represented by the product of two matrices $\mathcal{M}_G$ and $\mathcal{M}_F$, that is

$$\mathcal{M}_{GF} = \mathcal{M}_G \mathcal{M}_F.$$

Hyperbolic scator addition can be represented by the sum of two matrices since the sum is evaluated by the sum of each component. Let the matrix $\mathcal{M}_G$ be the sum of two scators be represented by the sum of the matrices $\mathcal{M}_A$ and $\mathcal{M}_B$, then

$$\mathcal{M}_{GF} = (\mathcal{M}_A + \mathcal{M}_B)\,\mathcal{M}_F.$$

However, matrix product is distributive over addition, thus

$$\mathcal{M}_{GF} = \mathcal{M}_A\mathcal{M}_F + \mathcal{M}_B\mathcal{M}_F.$$

But we have already proved that the scator product is not distributive over addition. Therefore, the scator product cannot be isomorphic to matrix multiplication. □

The concept of modulus of a complex number can be extended to hypercomplex numbers by taking the $n$-th root of the absolute value of the characteristic determinant [4, ch.2.1.6]. It is not possible to extend this procedure to this case because the characteristic matrix does not contain all the information involved in the product of two arbitrary scators.

## 6. Lack of Distributivity

The difference between the product $(\alpha+\beta)\,\gamma$ and the sum of the products $\alpha\gamma+\beta\gamma$ is a measure of the lack of distributivity. The product $(\alpha+\beta)\,\gamma$ is

$$(\alpha+\beta)\,\gamma = (a_0+b_0)\,g_0\left(1+\frac{(a_1+b_1)\,g_1}{(a_0+b_0)\,g_0}\right)\left(1+\frac{(a_2+b_2)\,g_2}{(a_0+b_0)\,g_0}\right)$$
$$\times\left(1;\frac{\left(\frac{(a_1+b_1)}{(a_0+b_0)}+\frac{g_1}{g_0}\right)}{\left(1+\frac{(a_1+b_1)g_1}{(a_0+b_0)g_0}\right)},\frac{\left(\frac{(a_2+b_2)}{(a_0+b_0)}+\frac{g_2}{g_0}\right)}{\left(1+\frac{(a_2+b_2)g_2}{(a_0+b_0)g_0}\right)}\right),$$

while the product $\alpha\gamma$ is

$$\alpha\gamma = a_0 g_0 \left(1 + \frac{a_1 g_1}{a_0 g_0}\right)\left(1 + \frac{a_2 g_2}{a_0 g_0}\right)\left(1; \frac{\left(\frac{a_1}{a_0} + \frac{g_1}{g_0}\right)}{\left(1 + \frac{a_1 g_1}{a_0 g_0}\right)}, \frac{\left(\frac{a_2}{a_0} + \frac{g_2}{g_0}\right)}{\left(1 + \frac{a_2 g_2}{a_0 g_0}\right)}\right),$$

and the product $\beta\gamma$ is

$$\beta\gamma = b_0 g_0 \left(1 + \frac{b_1 g_1}{b_0 g_0}\right)\left(1 + \frac{b_2 g_2}{b_0 g_0}\right)\left(1; \frac{\left(\frac{b_1}{b_0} + \frac{g_1}{g_0}\right)}{\left(1 + \frac{b_1 g_1}{b_0 g_0}\right)}, \frac{\left(\frac{b_2}{b_0} + \frac{g_2}{g_0}\right)}{\left(1 + \frac{b_2 g_2}{b_0 g_0}\right)}\right).$$

### 6.1. Scalar part

The difference between the product with the sum and the sum of the products $[(\alpha + \beta)\gamma - (\alpha\gamma + \beta\gamma)]_0$ for the scalar part is

$$\begin{aligned}&\left((a_0 + b_0) g_0 + (a_1 + b_1) g_1 + (a_2 + b_2) g_2 + \frac{(a_1 + b_1) g_1 (a_2 + b_2) g_2}{(a_0 + b_0) g_0}\right)\\ &- \left(a_0 g_0 + a_1 g_1 + a_2 g_2 + \frac{a_1 g_1 a_2 g_2}{a_0 g_0}\right)\\ &- \left(b_0 g_0 + b_1 g_1 + b_2 g_2 + \frac{b_1 g_1 b_2 g_2}{b_0 g_0}\right).\end{aligned}$$

This expression simplifies to

$$\left(\frac{(a_1 + b_1)(a_2 + b_2)}{(a_0 + b_0)} - \frac{a_1 a_2}{a_0} - \frac{b_1 b_2}{b_0}\right)\frac{g_1 g_2}{g_0}.$$

The term within brackets can be factored as

$$\left(\frac{(b_0 a_1 - a_0 b_1)(a_0 b_2 - b_0 a_2)}{(a_0 + b_0) a_0 b_0}\right)\frac{g_1 g_2}{g_0}.$$

### 6.2. Director component 1

The product of the sum for the first component is

$$[(\alpha + \beta)\gamma]_1 = (a_0 + b_0) g_0 \left(1 + \frac{(a_2 + b_2) g_2}{(a_0 + b_0) g_0}\right)\left(\frac{(a_1 + b_1)}{(a_0 + b_0)} + \frac{g_1}{g_0}\right)$$

that may be expanded as

$$\begin{aligned}&[(\alpha + \beta)\gamma]_1\\ &\quad = g_0 (a_1 + b_1) + (a_0 + b_0) g_1 + \frac{(a_2 + b_2) g_1 g_2}{g_0} + \frac{(a_1 + b_1)(a_2 + b_2) g_2}{(a_0 + b_0)}.\end{aligned}$$

While the product $[\alpha\gamma]_1$ for the first component is

$$\begin{aligned}[\alpha\gamma]_1 &= a_0 g_0 \left(1 + \frac{a_2 g_2}{a_0 g_0}\right)\left(\frac{a_1}{a_0} + \frac{g_1}{g_0}\right)\\ &= a_1 g_0 + a_0 g_1 + \frac{a_1 a_2 g_2}{a_0} + \frac{g_1 g_2 a_2}{g_0}\end{aligned}$$

and a similar result holds for $[\beta\gamma]_1$ with the substitution $a \to b$. The difference for the first component $[(\alpha+\beta)\,\gamma - (\alpha\gamma+\beta\gamma)]_1$ is then

$$\left(g_0\,(a_1+b_1)+(a_0+b_0)\,g_1+\frac{(a_2+b_2)\,g_1g_2}{g_0}+\frac{(a_1+b_1)\,(a_2+b_2)\,g_2}{(a_0+b_0)}\right)$$
$$-\left(a_1g_0+a_0g_1+\frac{a_1a_2g_2}{a_0}+\frac{g_1g_2a_2}{g_0}\right)$$
$$-\left(b_1g_0+b_0g_1+\frac{b_1b_2g_2}{b_0}+\frac{g_1g_2b_2}{g_0}\right).$$

This expression simplifies to

$$\left(\frac{(a_1+b_1)\,(a_2+b_2)}{(a_0+b_0)}-\frac{a_1a_2}{a_0}-\frac{b_1b_2}{b_0}\right)g_2$$

that may be factored as

$$\left(\frac{(b_0a_1-a_0b_1)\,(a_0b_2-b_0a_2)}{(a_0+b_0)\,a_0b_0}\right)g_2.$$

### 6.3. Director component 2

The difference $[(\alpha+\beta)\,\gamma - (\alpha\gamma+\beta\gamma)]_2$ for the second component follows a similar procedure that gives

$$\left(\frac{(b_0a_1-a_0b_1)\,(a_0b_2-b_0a_2)}{(a_0+b_0)\,a_0b_0}\right)g_1.$$

*Remarks* 6.1. Given scators $\alpha,\beta,\gamma$ with 1+2 components the distributivity difference is

$$(\alpha+\beta)\,\gamma-[(\alpha\gamma)+(\beta\gamma)] = \frac{(b_0a_1-a_0b_1)\,(a_0b_2-b_0a_2)}{(a_0+b_0)\,a_0b_0}\left(\frac{g_1g_2}{g_0};g_2,g_1\right). \quad (6.1)$$

Let us evaluate some particular cases:

(1) If the scators $\alpha$ and $\beta$ have only one and the same non-vanishing director component, say the second component, then the above expression is zero and the product is distributive over addition.

(2) If the scators $\alpha$ and $\beta$ have only one director component but not the same one, the above expression is different from zero, for example if $a_2=b_1=0$,

$$(\alpha+\beta)\,\gamma-[(\alpha\gamma)+(\beta\gamma)] = \frac{a_1b_2}{(a_0+b_0)}\left(\frac{g_1g_2}{g_0};g_2,g_1\right).$$

(3) If the scator $\gamma=\{g_0,g_1,0\}$ has only one component, then

$$(\alpha+\beta)\,\gamma-[(\alpha\gamma)+(\beta\gamma)] = \frac{(b_0a_1-a_0b_1)\,(a_0b_2-b_0a_2)}{(a_0+b_0)\,a_0b_0}\,(0;0,g_1)\,.$$

## 7. Composition of Velocities in A Deformed Lorentz Metric

We now examine one of the possible applications of real or hyperbolic scator algebra. In the canonical description of special relativity, the Lorentz transformations establish the operations required to transform time-space between inertial frames. The separation between events is chosen as the square root of a quadratic form. The signature of this form together with a Minkowskian system of coordinates completes the framework [16]. Einstein's theorem of addition of velocities is a consequence of this approach. Amendments to the Lorentz invariant have been proposed on different grounds, in particular regarding a fundamental length at the Planck scale [1, 12] and deformed metrics depending on the energy and nature of interactions [3]. The composition of velocities in special relativity ought to be consistent with the two fundamental postulates, the constancy of the speed of light and the equivalence of all observers in free inertial motion.

There is liberty regarding the choice of mathematical structure selected in order to model physical phenomena. Poincaré expressed this idea, according to Carnap, in the following statement: "... the physicist is free to ascribe to physical space any one of the mathematically possible geometrical structures, provided he makes suitable adjustments in the laws of mechanics and optics and consequently in the rules for measuring length." [15]. In accordance with this view, it should be possible to produce a different set of rules to perform an operation, say the product. These rules cannot be arbitrarily set, but have to fulfill the constraints of the geometrical structure according to the phenomena that are being modeled or described. The procedure for measuring magnitudes should also be altered, that is, the metric modified. Physical meaning can be ascribed to this operation, in this particular proposal, the composition of velocities between inertial frames. Let us follow these guidelines.

Allow for (2.1a)-(2.1c) to be the new set of product rules. Consider the quotient of the scator director components $a_1$ and $a_2$ over the scalar component $a_0$ represent the velocity components (divided by the real constant $c$) in the directions 1 and 2, that is $\beta_{a1} \equiv \frac{v_{a1}}{c}, \beta_{a2} \equiv \frac{v_{a2}}{c}$. The velocity scator is then $\frac{\overset{o}{\alpha}}{a_0} = (1; \beta_{a1}, \beta_{a2})$. In physics, the overset $\{\overset{o}{\alpha}\}$ symbol is used to represent scator quantities. Similarly, the velocity scators $\overset{o}{\beta}$ and $\overset{o}{\gamma}$ are $\frac{\overset{o}{\beta}}{b_0} = (1; \beta_{b1}, \beta_{b2})$ and $\frac{\overset{o}{\gamma}}{g_0} = (1; \beta_{g1}, \beta_{g2})$ respectively. Let the composition of velocities be represented by the product operation $\overset{o}{\gamma} = \overset{o}{\alpha}\overset{o}{\beta} = (g_0; g_1, g_2)$. The velocity component in direction 1 is

$$\beta_{g1} = \frac{g_1}{g_0} = \frac{\left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)}{\left(1 + \frac{a_1 b_1}{a_0 b_0}\right)} = \frac{(\beta_{a1} + \beta_{b1})}{(1 + \beta_{a1}\beta_{b1})}.$$

The composition of velocities in direction 2 is

$$\beta_{g2} = \frac{g_2}{g_0} = \frac{\left(\frac{a_2}{a_0} + \frac{b_2}{b_0}\right)}{\left(1 + \frac{a_2 b_2}{a_0 b_0}\right)} = \frac{(\beta_{a2} + \beta_{b2})}{(1 + \beta_{a2}\beta_{b2})}.$$

From these results, we have recently established the generalization for the composition of velocities to three dimensions [8]. The constraints imposed on the mathematical structure are that the product operation forms a commutative group. These constraints are established on the following grounds:

- Closure - the composition of velocities should be an admissible velocity.
- Neutral - a relative velocity frame should exist that does not alter the velocity of the object.
- Inverse - there should exist a frame for any event where the object is at rest.
- Associativity - the composition of velocities should fulfill the reciprocity principle [2]. The reciprocity principle regarding velocities may be stated as follows: Let the velocity of an inertial frame $\mathcal{F}_{obj}$ relative to another inertial reference frame $\mathcal{F}_{lab}$ be $\mathbf{u}'$, then reciprocally, the velocity of $\mathcal{F}_{lab}$ relative to $\mathcal{F}_{obj}$ is $-\mathbf{u}'$.
- Commutativity - the composition of velocities should fulfill the reciprocity principle.

Let the definition of magnitude be given by (2.3). This magnitude definition is identical to the Lorentz metric $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ in 1+1 dimensions (one spatial dimension). It also approaches the Lorentz metric in the paraxial regime. Departures arise in two or more spatial dimensions, that is, when the Thomas precession comes into play [8]. The equation for constant metric $m$ is given by $\sqrt{c^2t^2 - x^2 - y^2 + \frac{x^2y^2}{c^2t^2}} = m$; In polynomial form with $c = 1$,

$$t^4 - \left(m^2 + x^2 + y^2\right) t^2 + x^2y^2 = 0$$

For a light-like event, the constant $m$ is zero. The light curves for constant time $t$ are squares with side equal to $2t$. The light events surface is composed by two tetragonal pyramids joined by their vertices at the origin. A plot of the light pyramid as well as the light cone in Minkowski space is shown in figure 1.

The restricted space condition establishes the function domain, which, in the special relativity terminology, corresponds to the admissible velocity. For director components larger than the scalar components, spatial-like events are described. Hypothetical particles within this realm have been named tachyons. For director components smaller than the scalar components, time-like events are described. These are the events within the light pyramid accessible to an observer located at the vertex of the dipyramid.

The proposed framework is in accordance with the constancy of the speed of light limit regardless of the motion of the source. An important feature of the present proposal is that the composition of velocities forms a commutative group. The velocity reciprocity principle is then immediately

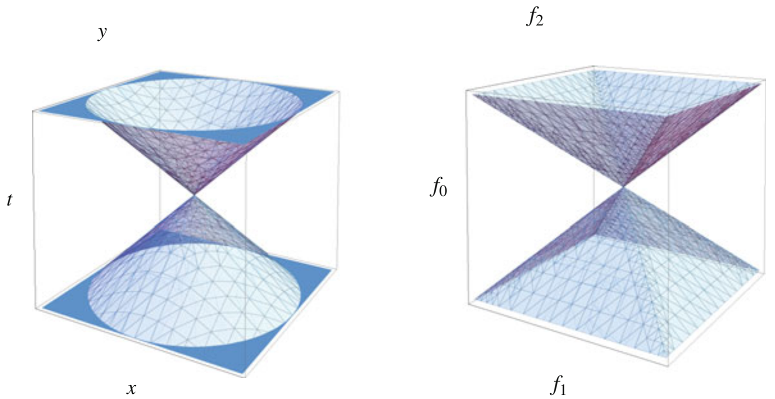


Figure 1. Minkowski's light cone (left) and light pyramid (right) obtained with the real scator metric. The vertical axis represents the time axis and the other orthogonal axes represent two spatial dimensions. The velocity at any point in the pyramid surface is always $c$ as may be seen from the scator metric definition.

fulfilled. Paradoxes that involve the composition of non-colinear velocities, such as the Mocanu paradox [13], do not arise in this Lorentz deformed metric space. Furthermore, there is no Thomas rotation in this scheme.

The geometric significance of zero divisors can be seen as follows: Given $\alpha = (a_0; a_1, a_2)$ an invertible scator, a zero divisor $\beta = (b_0; b_1, b_2)$ is obtained if $\beta = a_0 b_0 \left( \frac{1}{a_0}; -\frac{1}{a_1}, -\frac{1}{a_2} \right)$ due to conditions (2.8). Scator $\beta$ has then reciprocal slopes to scator $\alpha$ in the planes $a_0, a_1$ and $a_0, a_2$. If the scator $\alpha$ lies in the interior of the light pyramid, scator $\beta$ must lie outside the pyramid and viceversa. Phrased in relativity terms, the product of a temporal-like event with a spatial-like event is required to obtain a zero product.

If $\alpha$ is non invertible, that is, from (2.9a) and (2.9b), $\alpha = a_0 (1; \pm 1, a_2)$ or $\alpha = a_0 (1; a_1, \pm 1)$, then $\beta = b_0 (1; \mp 1, b_2)$ for the former scator or $\beta = b_0 (1; b_1, \mp 1)$ for the latter one. Therefore, $\alpha$ lies on the boundary of the light pyramid and $\beta$ lies on the opposite boundary slope. In relativity terms, the product of two light-like events with opposite sense in a given projection is zero.

## 8. Conclusions

We have presented an algebra structure in $\mathbb{R}^3$ with componentwise addition and novel product operation that has zero divisors. We have explicitly characterized these zero divisors. If these divisors are excluded, both, the addition and product operations satisfy commutative group properties. However, the product does not distribute over addition. The first scator component behaves like a real scalar whereas the two remaining components are symmetrical and exhibit a behaviour similar to quantities with direction. The construction can be generalized to any dimension. Note that in the two dimensional case, when one director component is omitted, distributivity holds and the corresponding algebra is the hyperbolic complex numbers. The product of two scators $\alpha = (a_0; a_1, a_2, \ldots, a_n)$ and $\beta = (b_0; b_1, b_2, \ldots, b_n)$ with arbitrary dimension $n$ is then defined by $\gamma = \alpha\beta = (g_0; g_1, g_2, \ldots, g_n)$, where the scalar component of the product is

$$g_0 = a_0 b_0 \prod_{k=1}^{n} \left(1 + \frac{a_k b_k}{a_0 b_0}\right) \tag{8.1}$$

and the j$^{\text{th}}$ director component is

$$g_j = a_0 b_0 \prod_{k \neq j}^{n} \left(1 + \frac{a_k b_k}{a_0 b_0}\right) \left(\frac{a_j}{a_0} + \frac{b_j}{b_0}\right). \tag{8.2}$$

The identity between the norm of a product and the product of the norms described in page 3 extended to $n$ dimensions gives rise to Lagrange's and higher order identities [9].

Complex numbers are to Euclidean geometry as hyperbolic numbers are to Minkowski space-time special relativity [4]. However, just as it occurs in Euclidean space, a system of hypercomplex numbers extended beyond the $1 + 1$ dimensional framework, and with nice algebra properties, has been elusive. It is thus natural to extend the $1 + 2$ dimensional hypercomplex system presented here to a deformed Lorentz-Minkowski time-space. This approach, as we have shown, gives a composition of velocities rule that is consistent with the fundamental relativity postulates.

This algebra is likely to find applications in models that require physical quantities in a restricted space that should fulfill group properties in an arbitrary number of dimensions. Another interesting application, currently being developed, is the quadratic iterated mapping with 1+2 real scators [18]. They exhibit a fractal structure in contrast with the smooth boundary generated by hyperbolic numbers.

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