Composition of physical quantities in one dimension: Group-theoretic differentiable functions

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We show that any group-theoretic differentiable operation in an open interval of real numbers is isomorphic to the usual addition of real numbers. Given the composition law, it is possible to establish the transformation relation. Alternatively, given a transformation, it is possible to obtain the composition relation in terms of the new variable. We show that some well known cases such as entropy and the relativistic addition of parallel velocities are included in this general framework. The composition rules for a wide variety of phenomena ranging from electrical circuits to thermodynamic systems are treated in a unified way.
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I. INTRODUCTION

There are many physical quantities that obey group properties. In many cases, it is possible to establish that a certain physical quantity is associative or commutative even if the specific composition relation is not known. Abrupt temporal or spatial changes in physical variables are in most cases not possible without violating fundamental physical principles such as relativity or requiring unattainable conditions such as infinite potentials.[1] Therefore, many physical quantities are time and space differentiable, and thus differentiable with respect to the variables of the composition rule.

Composition in electrical circuits consists in the addition of electrical components such as resistors, capacitors, or inductances either in series or parallel. In thermodynamics composition consists in merging several systems. In mechanics, the composition of masses or velocities between different reference frames is common. In the framework that we will discuss, the variable to be composed needs to be a real scalar quantity. Care should be taken with vector quantities, such as velocities, either by doing an analysis in one dimension or ensuring that each component is treated independently.

A connected non-compact one-dimensional Lie group is isomorphic to $\mathbb{R}$ with the usual addition operation. The proof of this theorem is usually given within the Lie algebra formalism. However, it is possible as will be shown here, to give a self-contained proof of this theorem by invoking only basic undergraduate calculus. The proof provides the relation between transformations and composition rules. The theorem can then be linked to the composition of physical quantities in one dimension.

In Sec. II, we show that any differentiable group law in an open interval of real numbers is isomorphic to the additive group of real numbers. An explicit expression for the transformation in terms of the composition rule will be given. Two broad problems, described in Secs. III and IV, can be treated with the present formalism.

II. GROUP LAWS IN OPEN INTERVALS

Suppose that $I=(a,b)\subseteq\mathbb{R}$ in an open interval with a group operation, that is, with a function $F\colon I\times I\to I$, such that

(1) If $x,y,z\in I$, then $F(F(x,y),z)=F(x,F(y,z))$.
(2) There is an element $e\in I$ such that $F(x,e)=x=F(e,x)$, for all $x\in I$. The element e is the *identity* of the group I.
(3) If $x\in I$, there is an element $i(x)\in I$ such that $F(x,i(x))=e=F(i(x),x)$. The element $i(x)$ is the *inverse* of the element x.

The composition rule is often written as a binary operation $F(x,y)=x\oplus y$ in order to view it as a generalization of the usual sum operation. However, it is useful to write the operation as a function $F(x,y)$ of two real variables to be able to write explicitly its derivatives. The main result in this context is the following proposition:

If $F\colon I\times I\to I\subseteq\mathbb{R}$ is a group law in an open interval I and has continuous partial derivatives, there exists a differentiable isomorphism $\lambda\colon I\to(\mathbb{R},+)$; that is, λ is bijective and for all $x,y\in I$,

$$\lambda(x\oplus y)=\lambda(x)+\lambda(y). \quad (1)$$

To find an explicit expression for λ, suppose for a moment that it exists and has the required properties. In particular, $\lambda(x\oplus y)=\lambda(x)+\lambda(y)$ can be written in terms of F as

$$\lambda(F(x,y))=\lambda(x)+\lambda(y). \quad (2)$$

Because we have assumed that λ is differentiable, we can differentiate both sides with respect to x and obtain

$$\lambda'(F(x,y))\frac{\partial F(x,y)}{\partial x}=\lambda'(x). \quad (3)$$

We set $x=e$ (the identity of I) and find

$$\lambda'(y)F_1(e,y)=\lambda'(e), \quad (4)$$

because $F(e,y)=y$. We let F_1 denote the partial derivative with respect to the first variable $F_1(e,y)\equiv\partial F(x,y)/\partial x|_{x=e}$. We then solve for $\lambda'(y)$ and integrate to find

$$\lambda(y)=\int_e^y\frac{\lambda'(e)}{F_1(e,\tau)}d\tau. \quad (5)$$

Two points need to be verified. First, we must show that $F_1(e,\tau)$ is never zero for all $\tau \in I$. The second point is that we did not include an additive constant of integration, because we want λ to be a homomorphism. Therefore, λ must map the identity e of I to the identity 0 of $\mathbb{R}$, that is, $\lambda(e)=0$ and Eq. (5) takes this condition into account. Note that $\lambda'(e)$ is a constant, and we will see that this constant is the required rescaling factor.

We now show that for all $\tau \in I$, $F_1(e,\tau) > 0$. We differentiate the associativity relation $F(F(x,y),z) = F(x,F(y,z))$ with respect to x,

$$F_1(F(x,y),z)F_1(x,y) = F_1(x,F(y,z)), \tag{6}$$

and set $x=e$ (the identity of I) and obtain

$$F_1(y,z)F_1(e,y) = F_1(e,F(y,z)). \tag{7}$$

Therefore, if $F_1(e,y)=0$ for some $y \in I$, then $F_1(e,F(y,z)) = 0$ for all $z \in I$. We choose $z=i(y)$ and obtain $F_1(e,e)=0$, which cannot be true because $F(x,e)=x$ for all x. Thus, $\partial F(x,e)/\partial x = F_1(x,e) = dx/dx = 1$. In particular, $F_1(e,e)=1$. It follows that $F_1(e,y)\neq 0$ for all $y\in I$, and because for $y=e$, we have $F_1(e,e)=1$, and F_1 is continuous. Thus, $F_1(e,y)$ must always be positive.

From Eq. (4), the function $1/F_1(e,\tau)$ is continuous in τ, and hence is integrable. Thus we can define

$$\lambda(x) \equiv \int_e^x \frac{d\tau}{F_1(e,\tau)}. \tag{8}$$

By the fundamental theorem of calculus, it follows that λ is differentiable and

$$\lambda'(x) = \frac{1}{F_1(e,x)}, \tag{9}$$

where in particular $\lambda'(e)=1$.

By using the definition (8) for λ we can show that $\lambda(F(x,y)) = \lambda(x)+\lambda(y)$ and λ: $I \to \mathbb{R}$ are bijective. For the first part, we fix $y\in I$ and consider the derivative with respect to x of both functions $\lambda(F(x,y))$ and $\lambda(x)+\lambda(y)$, respectively. From Eq. (9), the derivative of the first function is

$$\lambda'(F(x,y))F_1(x,y) = \frac{F_1(x,y)}{F_1(e,F(x,y))}. \tag{10}$$

By differentiating the second function, we obtain Eq. (9). We require that Eqs. (10) and (9) are equal, that is

$$\frac{F_1(x,y)}{F_1(e,F(x,y))} = \frac{1}{F_1(e,x)}, \tag{11}$$

or equivalently,

$$F_1(x,y)F_1(e,x) = F_1(e,F(x,y)). \tag{12}$$

This result is the associativity condition (7), where x,y,z is replaced by e,x,y. We have shown that the functions $\lambda(F(x,y))$ and $\lambda(x)+\lambda(y)$ differ by an additive constant, and because they are equal for $x=e$, it follows that the additive constant should be zero, and so we have the required equality

$$\lambda(F(x,y)) = \lambda(x)+\lambda(y). \tag{13}$$

To show that λ is bijective, observe that because $\lambda'(y) = 1/F_1(e,y) > 0$, λ is increasing and therefore is injective. To show surjectivity, observe that because λ is continuous, $\lambda(I)$ is an interval of $\mathbb{R}$. Choose $x\neq e$ in I and observe that because

$$\lambda(x)+\lambda(i(x)) = \lambda(x\oplus i(x)) = \lambda(e) = 0, \tag{14}$$

$\lambda(x)$ and $\lambda(i(x))$ are both nonzero and have opposite signs. For all positive integers n,

$$\lambda(\overbrace{x\oplus\cdots\oplus x}^{n}) = n\lambda(x), \tag{15}$$

and

$$\lambda(\overbrace{i(x)\oplus\cdots\oplus i(x)}^{n}) = n\lambda(i(x)). \tag{16}$$

Thus, when $n\to\infty$, one of the two sequences goes to $+\infty$ and the other to $-\infty$, and because $\lambda(I)$ is an interval, we must have that $\lambda(I)=\mathbb{R}$. Thus, we have shown that λ is bijective.

If the group law $x\oplus y = F(x,y)$ has continuous partial derivatives, the group is abelian. This property follows from the isomorphism given in the proposition because the group of real numbers under the usual addition operation is commutative. The function λ determines the group operation in $(I,\oplus)$ because it is bijective. Its inverse λ^{-1} satisfies

$$x\oplus y = F(x,y) = \lambda^{-1}(\lambda(x)+\lambda(y)). \tag{17}$$

III. TRANSFORMATIONS FROM COMPOSITION EXPRESSIONS

A. Entropy and thermodynamic probability states

We may think of λ: $(I,\oplus)\to(\mathbb{R},+)$ as a logarithm, because this is the transformation function when the interval is $I=(0,\infty)$ and the operation in I is the usual product of positive real numbers $F(x,y)=xy$, with the identity $e=1$. In this case $F_1(1,x)=x$, and the proposition implies that

$$\lambda(x) = \int_1^x \frac{d\tau}{F_1(1,\tau)} = \int_1^x \frac{d\tau}{\tau} = \ln(x). \tag{18}$$

This result is reminiscent of the relation between entropy and thermodynamic probability. Recall that the thermodynamic probability is the number of microstates corresponding to a given macrostate.[2] It is a positive integer bounded between one and the total number of microstates. The result by Planck[3] involves two premises: the entropy of two systems equals the sum of the individual entropies, and the probability of independent states equals the product of the individual probabilities. The entropy composition law is the usual sum in the reals, and thus it forms an abelian group under the addition operation. The thermodynamic probability composition law is defined in terms of the product of real positive numbers. These two groups are therefore isomorphic with transformations given by the logarithm and its inverse, the exponential.

B. Relativistic addition of velocities

In the special theory of relativity in one dimension, the velocity of a particle is restricted to the interval of real numbers

$(-c, c)$, where c is the speed of light in vacuum, and the addition of velocities is given by

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + (v_1 v_2)/c^2}. \tag{19}$$

It can be verified that if $v_1, v_2 \in (-c, c)$, then $v_1 \oplus v_2 \in (-c, c)$, addition is associative, has an identity element $0 \in (-c, c)$, and inverses exist. In other words, the interval $(-c, c)$ is a *group* with the operation given by Eq. (19). This group structure for the relativistic composition law has been recognized either implicitly or explicitly by several authors.[4,5] For $I = (-c, c)$ and $F(v_1, v_2) = v_1 \oplus v_2 = (v_1 + v_2)/(1 + v_1 v_2/c^2)$, we have $e = 0$ and $F_1(0, \tau) = (c^2 - \tau^2)/c^2$. Hence from Eq. (8), the transformation of $\lambda_c\colon (-c, c) \to \mathbb{R}$ is

$$\lambda_c(v) = \int_0^v \frac{d\tau}{F_1(0,\tau)} = \frac{c}{2}\int_0^v \left(\frac{1}{c-\tau} + \frac{1}{c+\tau}\right) d\tau \tag{20a}$$

$$= \frac{c}{2}\ln\left(\frac{1 + v/c}{1 - v/c}\right) = c; \quad \operatorname{arctanh}(v/c). \tag{20b}$$

Recall that the hyperbolic tangent $\tanh\colon \mathbb{R} \to (-1, 1)$ is a bijection with inverse $\operatorname{arctanh}(x) = (1/2)\ln(1+x)/(1-x)$. Thus we recognize $\lambda_c(v)$ as a rescaling of the inverse hyperbolic tangent. Therefore, the inverse of λ_c is the rescaled hyperbolic tangent function

$$\lambda_c^{-1}(v) = c\tanh(v/c). \tag{21}$$

Moreover, it satisfies $\lambda_c^{-1}(v_1 + v_2) = \lambda_c^{-1}(v_1) \oplus \lambda_c^{-1}(v_2)$. The addition of velocities in Eq. (19) is thus given by the addition formula for the hyperbolic tangent

$$c\tanh((v_1 + v_2)/c) = c\left(\frac{\tanh(v_1/c) + \tanh(v_2/c)}{1 + \tanh(v_1/c)\tanh(v_2/c)}\right) = \frac{\lambda_c^{-1}(v_1) + \lambda_c^{-1}(v_2)}{1 + \dfrac{\lambda_c^{-1}(v_1)\lambda_c^{-1}(v_2)}{c^2}}, \tag{22}$$

a result usually described in terms of a pseudo-angle of rotation of the Lorentz transformations (see Ref. 6). The function λ_c^{-1} is an isomorphism from the group of real numbers with the usual sum, denoted by $(\mathbb{R}, +)$ to the group $(-c, c)$ with the addition $\oplus$ given by Eq. (19). A differentiable group operation in the interval $(-c, c)$ is thus necessarily equal, up to an isomorphism, to the Lorentz–Einstein addition of parallel velocities. This derivation is a generalization of the argument given by David Mermin.[7]

IV. COMPOSITION RELATION GIVEN THE TRANSFORMATION

If the transformation λ is known, then from Eq. (17) the composition may be readily obtained. From the identity $\lambda^{-1}\lambda(x) = I(x) = x$, we recover the composition law as $F(x, y) = \lambda^{-1}[\lambda(x) + \lambda(y)]$. For example, the inverse of the logarithm transformation $\lambda \to \ln$ is the exponential. The composition rule is then $F(x, y) = \lambda^{-1}[\lambda(x) + \lambda(y)] = \exp[\ln x + \ln y] = xy$, and hence the product rule is obtained. In the entropy example, the composition of two non-interacting thermodynamic systems requires that the total entropy equal the usual sum of (real positive) entropies. Given the exponential transformation, to obtain the thermodynamic probability from the entropy, the composite thermodynamic probability must be the product of the individual thermodynamic probabilities.

A. Inverse relation

The inverse relation between variables is common in physics. The inverse potential $V \propto 1/x$ gives rise to the familiar inverse quadratic forces in electromagnetism and gravitation. Composition rules for quantities of the form $1/x$ are also familiar in electrical circuits for parallel resistors or capacitors in series. This inverse relation is also observed in the classical equipartition principle $U_\beta = 1/\beta$, where the mean energy is U_β, $\beta = 1/kT$, k is Boltzmann's constant, and T is the temperature. Given this transformation, the composition rule is readily obtained from Eq. (17)

$$F(x, y) = \lambda^{-1}[\lambda(x) + \lambda(y)] = \left[\frac{1}{x} + \frac{1}{y}\right]^{-1} = \left[\frac{x+y}{xy}\right]^{-1} = \frac{xy}{x+y}. \tag{23}$$

This operation does not satisfy the group differentiable requirements. Thus, it is not surprising that $F_1(e, y)$ poses problems, because it should never be equal to zero. We propose the following formalism in the context of Lie groups to overcome these drawbacks. Let us consider the set $I = (\mathbb{R} - \{0\}) \cup \{\infty\}$, where we define the operation

$$x \oplus y = \begin{cases} \dfrac{xy}{x+y} & x \neq 0, y \neq 0, x \neq -y \\ \infty & x \neq 0, y \neq 0, x = -y \\ x & y = \infty \\ y & x = \infty. \end{cases} \tag{24}$$

It is straightforward to check that this operation is associative and commutative, ∞ is the identity, the inverse of x is $-x$ and $-\infty = \infty$. Thus $(I, \oplus)$ is an abelian group. This construction corresponds to a stereographic projection in one dimension. The mapping $\lambda\colon I \to (\mathbb{R}, +)$ defined by

$$\lambda(x) = \begin{cases} \dfrac{1}{x} & x \neq \infty \\ 0 & x = \infty, \end{cases} \tag{25}$$

is an isomorphism with inverse $\lambda^{-1}\colon (\mathbb{R}, +) \to I$

$$\lambda^{-1}(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ \infty & x = 0. \end{cases} \tag{26}$$

We define open neighborhoods of ∞ as images under λ^{-1} of open neighborhoods of $0 \in \mathbb{R}$ so that λ is a homeomorphism. If we write the derivative of the composition rule at the neutral as a limit without explicit evaluation

$$F_1(e, y) = \frac{d}{dx}\left(\frac{xy}{x+y}\right)\bigg|_{x=e} = \left(\frac{y}{x+y}\right)^2\bigg|_{x=e} = \lim_{e\to\infty}\left(\frac{y}{e+y}\right)^2, \tag{27}$$

the transformation λ can be consistently obtained from Eq. (8) evaluated in the limit

$$\lambda(y) = \lim_{e\to\infty} \int_e^y \frac{\lambda'(e)}{F_1(e,\tau)} d\tau = \lim_{e\to\infty} \int_e^y \frac{-e^{-2}}{\left(\frac{\tau}{e+\tau}\right)^2} d\tau, \quad (28)$$

because $\lambda'(e) = -e^{-2}$. The transformation is thus

$$\lambda(y) = \lim_{e\to\infty} \left\{ -\int_e^y \frac{(e+\tau)^2}{\tau^2 e^2} d\tau \right\} = -\int_\infty^y \frac{1}{\tau^2} d\tau = \frac{1}{y}. \quad (29)$$

This result shows that the $1/x$ transformation can be obtained as a limit case of the one-dimensional Lie group formalism.

We remark that the composition rule for two systems satisfying the equipartition principle with temperatures $\beta_1 = 1/kT_1$ and $\beta_2 = 1/kT_2$ is, according to Eq. (23),

$$U_\beta = \frac{U_{\beta 1} U_{\beta 2}}{U_{\beta 1} + U_{\beta 2}}. \quad (30)$$

For a quantum system, the mean oscillator energy U_P as a function of β is (Ref. 8)

$$U_P = \frac{1}{2} \hbar\omega \coth\left(\frac{\hbar\omega}{2}\beta\right). \quad (31)$$

This transformation leads to the mean oscillator energy composition rule

$$U_{P1} \oplus U_{P2} = \frac{(\hbar\omega/2)^2 + U_{P1} U_{P2}}{U_{P1} + U_{P2}}. \quad (32)$$

for $(\hbar\omega/2)\beta_1 + (\hbar\omega/2)\beta_2$. Equation (32) reduces to the classical equipartition transformation (30) in the limit $\hbar \to 0$.

V. CONCLUSIONS

We have shown that any differentiable group law in an open interval of real numbers is isomorphic to the additive group of real numbers. When the composition rule is known, it is possible to obtain the transformation into a group that composes with the real number addition rule. If the transformation between two quantities is known and one of them adds with the usual real addition rule, then the composition rule for the other variable can be obtained.

This procedure can be applied to a wide variety of phenomena. The condition is that the physical quantity should fulfill group differentiable properties, most importantly the associative and differentiable requirements. Examples we discussed include the entropy and thermodynamic probability, the relativistic addition of velocities, and $1/x$ potentials. In the last example, care should be taken to preserve the group structure. The differentiable condition is quite stringent. For example, spaces with metric of the form $F(x,y) = \sqrt[n]{x^n + y^n}$ are not differentiable at the origin $F_1(e,e)$, and thus the transformation λ cannot be obtained from the theorem.

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[1]M. Fernández-Guasti, "Attainable conditions and exact invariant for the time-dependent harmonic oscillator," J. Phys. A: Math. Gen. **39**, 11825–11832 (2006).

[2]J. S. Dugdale, *Entropy and Its Physical Meaning* (Taylor and Francis, London, 1996).

[3]M. Planck, *The Theory of Heat Radiation* (Dover, Mineola, 1991), p. 118.

[4]R. Penrose and W. Rindler, "Energy conservation as a basis of relativistic mechanics," Am. J. Phys. **33**, 55–59 (1965).

[5]S. Sonego and M. Pin, "Deriving relativistic momentum and energy," Eur. J. Phys. **26**(1), 33–45 (2005).

[6]J. L. Synge, *Relativity: The Special Theory* (North Holland, Amsterdam, 1972), p. 125.

[7]N. David Mermin, "Relativity without light," Am. J. Phys. **52**, 119–124 (1984).

[8]L. de la Peña, A. Valdés-Hernández, and A. M. Cetto, "Statistical consequences of the zero-point energy of the harmonic oscillator," Am. J. Phys. **76**, 947–955 (2008).