

Alternative realization for the composition of relativistic velocities

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ABSTRACT

The reciprocity principle requests that if an observer, say in the laboratory, sees an event with a given velocity, another observer at rest with the event must see the laboratory observer with minus the same velocity. The composition of velocities in the Lorentz-Einstein scheme does not fulfill the reciprocity principle because the composition rule is neither commutative nor associative. In other words, the composition of two non-collinear Lorentz boosts cannot be expressed as a single Lorentz boost but requires in addition a rotation. The Thomas precession is a consequence of this composition procedure. Different proposals such as gyro-groups have been made to fulfill the reciprocity principle.

An alternative velocity addition scheme is proposed consistent with the invariance of the speed of light and the relativity of inertial frames. An important feature of the present proposal is that the addition of velocities is commutative and associative. The velocity reciprocity principle is then immediately fulfilled. This representation is based on a transformation of a hyperbolic scator algebra. The proposed rules become identical with the special relativity addition of velocities in one dimension. They also reduce to the Galilean transformations in the low velocity limit. The Thomas gyration needs to be revised in this nonlinear realization of the special relativity postulates. The deformed Minkowski metric presented here is compared with other deformed relativity representations.

1. INTRODUCTION

There is liberty regarding the choice of mathematical structure selected in order to describe and predict physical phenomena. Poincaré established this idea, according to Carnap, in the following terms “No matter what observational facts are found, the physicist is free to ascribe to physical space any one of the mathematically possible geometrical structures, provided he makes suitable adjustments in the laws of mechanics and optics and consequently in the rules for measuring length”.¹ From my point of view, within such a mathematical structure it is necessary to establish the relationships between the categories of the theory and the way the physical variables ought to be measured. Furthermore, the predictions and relationships between variables of the theory should be compared with observations. These observations are concomitant with the theory on two grounds, firstly the physical categories that the researcher conceives and decides to observe and secondly, the way these quantities are to be measured. The observations should be in accordance with the theory within experimental and theoretical limitations and provide feedback towards the appropriateness and refinement of the abstract models.

Let us translate these rather broad assertions to the specific problem of the velocities of material bodies in inertial frames. The addition of velocities in special relativity ought to be consistent with the two fundamental postulates, the constancy of the speed of light and the equivalence of all observers in free inertial motion. In order to derive Einstein’s theorem of addition of velocities, the separation between space - time events is chosen as the square root of a quadratic form. Then, the appropriate signature of this form together with a Minkowskian system of coordinates, leads to the Lorentz transformations.²

However, this choice of a quadratic form need not be unique in order to fulfill the fundamental postulates. To wit, it may be possible to state velocity composition relationships different from those derived from the Lorentz transformations, which may nonetheless be consistent with an invariant velocity of light and the relativity of inertial frames. The present communication explores one such possibility based on a transformation of a

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hyperbolic scator algebra. It should be recalled that amendments to the Lorentz invariant have been proposed on different grounds by several researchers, in particular regarding a fundamental length at the Planck scale^{3,4} and deformed metrics depending on the energy and nature of interactions.⁵ A highlight of the present proposal is the consistency with the reciprocity principle without further ado. The reciprocity principle, heuristically stated, requests that if a reference frame (an observer) sees another frame (an object) receding with velocity \mathbf{v} , then the object sees the observer receding at $-\mathbf{v}$. The gyrogroup theory⁶ regarding the composite-velocity reciprocity principle is compared with the present formalism.

In the next section, the new velocity scheme is introduced. Section three compares the present proposal with the special relativity and Galilean velocity transformations with emphasis on the similarities between the various approaches. Section four shows the relationship of sequential Lorentz boosts in orthogonal directions with the new proposal. In section five, the reciprocity principle is addressed. In section six, the Thomas rotation and the possibility of its absence are briefly discussed. In section seven, this proposal is discussed within the framework of a deformed metric formulation. Conclusions are drawn in the last section.

2. COMPOSITION OF VELOCITIES SCHEME

Firstly, consider the following proposal for the addition of velocities:

composition of velocities Let two velocity elements be $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$; $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The composition of these elements is given by

$$\mathbf{v} \circledast \mathbf{u} \equiv \left(\frac{v_1 + u_1}{1 + v_1 u_1 / c^2}, \frac{v_2 + u_2}{1 + v_2 u_2 / c^2}, \frac{v_3 + u_3}{1 + v_3 u_3 / c^2} \right) \quad (1)$$

where we have used the circled asterisk symbol to represent this operation.

Secondly, allow for a rule to measure the length of a velocity element:

magnitude The magnitude of an arbitrary velocity is defined as

$$\|\mathbf{u}\|_d \equiv c \left[1 - \prod_{j=1}^3 \left(1 - \frac{u_j^2}{c^2} \right) \right]^{\frac{1}{2}}, \quad (2)$$

where the positive square root is chosen. The sub index j stands for the velocity components in three dimensional space.

And finally, permit for a new definition of admissible velocity:

admissible velocity An admissible velocity is defined by a velocity element whose individual components are less or equal to the velocity of light in vacuum $u_j^2 < c^2$. Admissible velocities are elements in the restricted space $\mathbb{R}_t^3 = \{\mathbf{u} \in \mathbb{R}^3 : u_j^2 < c^2\}$.

2.1 operation properties

From the symmetry of the variables in the composition operation definition (1) it follows that this operation is commutative $\mathbf{v} \circledast \mathbf{u} = \mathbf{u} \circledast \mathbf{v}$. Let us then judge whether it is also associative. To this end, consider the successive addition of elements $\mathbf{w} \circledast (\mathbf{v} \circledast \mathbf{u}) = (w_1, w_2, w_3) \circledast [(v_1, v_2, v_3) \circledast (u_1, u_2, u_3)]$. Since the addition rule involves the sum of each component independently for orthogonal components, the coordinate projections may be worked out separately

$$w_j \circledast (v_j \circledast u_j) = w_j \circledast \frac{v_j + u_j}{1 + v_j u_j / c^2} = \frac{w_j + (v_j + u_j) (1 + v_j u_j / c^2)^{-1}}{1 + \frac{w_j}{c^2} (v_j + u_j) (1 + v_j u_j / c^2)^{-1}}$$

multiply and divide by $(1 + v_j u_j / c^2)$, then

$$w_j \circledast (v_j \circledast u_j) = \frac{w_j + v_j + u_j + \frac{w_j v_j u_j}{c^2}}{1 + \frac{v_j u_j + w_j v_j + w_j u_j}{c^2}}$$

The symmetrical way in which the three variables are involved is already suggestive of associativity. But let us not anticipate; factor $(1 + w_j v_j / c^2)$ and finally multiply and divide by $(1 + w_j v_j / c^2)^{-1}$ to obtain

$$w_j \otimes (v_j \otimes u_j) = \frac{(w_j + v_j) (1 + w_j v_j / c^2)^{-1} + u_j}{1 + \frac{u_j}{c^2} (w_j + v_j) (1 + w_j v_j / c^2)^{-1}} = (w_j \otimes v_j) \otimes u_j$$

The coordinate projections may then be grouped back in vector form to yield $\mathbf{w} \otimes (\mathbf{v} \otimes \mathbf{u}) = (\mathbf{w} \otimes \mathbf{v}) \otimes \mathbf{u}$. Therefore, the composition of velocities operation is also associative. The identity element is $\mathbf{0} = (0, 0, 0)$ and inversion is identical to vector inversion $-\mathbf{u} = -(u_1, u_2, u_3) = (-u_1, -u_2, -u_3) \forall \mathbf{u} \in \mathbb{R}^3$. It then follows that together with closure, *the velocity elements form a group under the proposed operation* (\mathbb{R}^3, \otimes) . An algebraic structure that embraces these properties, termed hyperbolic or real scator algebra, is presently being developed.

2.2 magnitude

Consider the magnitude of a three dimensional velocity element in the presently suggested framework. An admissible velocity requires that $u_j^2 < c^2$, this inequality may be written as $0 < 1 - u_j^2 / c^2$. If each component satisfies this relationship, their product also satisfies the inequality $0 < \prod (1 - u_j^2 / c^2)$; Multiplication of this expression times minus one, subsequently adding one on both sides and taking the positive square root gives $1 > (1 - \prod (1 - u_j^2 / c^2))^{\frac{1}{2}}$. The right hand side of this inequality times c is precisely the proposed magnitude definition (2). Therefore, the magnitude is upper bounded by the velocity of light for any admissible velocity components. The subspace of admissible velocities may then be equivalently written as

$$\mathbb{R}_t^3 = \{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\|_d < c\}. \quad (3)$$

The magnitude of an admissible velocity is always a real positive quantity. The squared magnitude of the velocity vector if the product is expanded is

$$\|\mathbf{u}\|_d^2 = (u_1^2 + u_2^2 + u_3^2) - \frac{1}{c^2} (u_1^2 u_2^2 + u_2^2 u_3^2 + u_3^2 u_1^2) + \frac{1}{c^4} u_1^2 u_2^2 u_3^2. \quad (4)$$

2.3 example

Consider an object whose relative velocity with respect to an observer is three quarters the speed of light $\mathbf{u} = \frac{3}{4}c\hat{e}_1$. Let the observer move with respect to another reference frame, say the laboratory frame with an orthogonal velocity again three quarters of c , namely $\mathbf{v} = \frac{3}{4}c\hat{e}_2$. The composition of these velocities in the proposed scheme (1) gives the velocity of the object as seen from the laboratory frame

$$\mathbf{u}' = \frac{3}{4}c\hat{e}_1 + \frac{3}{4}c\hat{e}_2,$$

as shown in figure 1. It takes a while to recognize that the velocity element with components three quarters the velocity of light in orthogonal directions has a magnitude in this new scheme equal to

$$\|\mathbf{u}\|_d = \sqrt{u_1^2 + u_2^2 - \frac{1}{c^2} u_1^2 u_2^2} = \sqrt{\frac{9}{16} + \frac{9}{16} - \frac{81}{256}}c = \sqrt{\frac{207}{256}}c \approx 0.9c$$

rather than the usual SR vector magnitude $\|\mathbf{u}\| = \sqrt{18/16}c \approx 1.06c!$ The central difference regarding magnitudes is twofold, on the one hand, the magnitude in the present proposition exhibits extra crossed terms involving the products of orthogonal velocity components. On the other hand, admissible velocities in Einstein's addition theorem require that $\sum u_j^2 < c^2$ whereas in this proposal, admissible velocities impose the less restrictive condition $u_j^2 < c^2$.

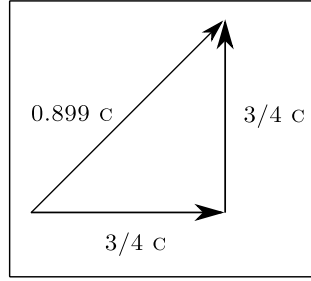


Figure 1. Composition of perpendicular velocities in the *sequentially invariant relativity* (SIR) framework.

3. COMPARISON WITH ORTHODOX VELOCITY TRANSFORMATIONS

Recall that the addition of velocities, according to the Lorentz transformations, may be expressed in vector form as⁷

$$\mathbf{v} \oplus \mathbf{u} \equiv \frac{1}{1 + \mathbf{v} \cdot \mathbf{u}/c^2} \left(\mathbf{v} + \mathbf{u} + \frac{1}{c^2} \frac{\gamma_v}{1 + \gamma_v} (\mathbf{v} \times (\mathbf{v} \times \mathbf{u})) \right) \quad (5)$$

where \mathbf{v} , \mathbf{u} are admissible velocities in special relativity (SR). The Lorentz factor being $\gamma_v = (1 - \mathbf{v} \cdot \mathbf{v}/c^2)^{-\frac{1}{2}}$. The vector magnitudes are given by the usual form $\|\mathbf{v}\| \equiv \sqrt{\mathbf{v} \cdot \mathbf{v}}$, $\|\mathbf{u}\| \equiv \sqrt{\mathbf{u} \cdot \mathbf{u}}$ and admissible velocities are defined by $\mathbb{R}_c^3 = \{\mathbf{v}, \mathbf{u} \in \mathbb{R}^3 : \|\mathbf{v}\|, \|\mathbf{u}\| < c\}$. Notice that the magnitude of the velocity $\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2$ has the same Euclidean quadratic form either in special relativity or the Galilean framework*. In contrast, the magnitude of the velocity vector in the present proposal does not have an Euclidean form as may be seen from (4).

3.1 one dimensional relativistic case

Consider the special case of parallel velocities in one dimension, say in the $j = 1$ direction [2, ch.V-4, p.125]. The composite velocity obtained from the proposed addition operation ((1)) and from the Lorentz transformations (5) become identical, $\mathbf{v} \circledast \mathbf{u} = \mathbf{v} \oplus \mathbf{u} = (v_1 + u_1) (1 + v_1 u_1/c^2)^{-1}$. In this particular case, the magnitude introduced in e. (4) reduces to the usual single vector component magnitude $\|\mathbf{u}\|_t = \|(u_1, 0, 0)\| = \sqrt{u_1^2}$. Furthermore, the definition of admissible velocity is then the same in either scheme. For admissible velocities, the composite velocity of any one component is well established to be upper bounded by the velocity of light. Therefore, *in the one dimensional case, the proposed composition rule, magnitude and admissible velocity definitions are identical to those obtained from the Lorentz transformations.*

3.2 low velocity limit

Let us now turn to the three dimensional problem in the low velocity limit. The addition of velocities presented in equation (1) becomes $\mathbf{v} \circledast \mathbf{u} \rightarrow \mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$ and it is then equal to the Galilean transformation of velocities. Furthermore, if the velocity is small compared with c , the magnitude (2) becomes the usual Euclidean magnitude

$$\lim_{u_j \ll c} \|\mathbf{u}\|_d = \left(\sum_{j=1}^3 u_j^2 \right)^{\frac{1}{2}} = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

Hence, *the proposed composition operation and its magnitude approach the Galilean addition of velocities and the vector magnitude in the low velocity limit.*

*Only when the fourth variable is introduced in special relativity is the four vector metric hyperbolic.

3.3 constancy of the speed of light

Let us appraise the limit where one of the velocities involved approaches the velocity of light. According with the magnitude definition (2), a velocity whose magnitude is c necessarily has to have at least one projection whose value is c . Consider the velocity $\mathbf{u} = (u_1 \rightarrow c, u_2, u_3)$ with the component in the \hat{e}_1 direction approaching c . The magnitude is

$$\lim_{u_1 \rightarrow c} \|\mathbf{u}\|_d = c \left[1 - \lim_{u_1 \rightarrow c} \left(1 - \frac{u_1^2}{c^2} \right) \left(1 - \frac{u_2^2}{c^2} \right) \left(1 - \frac{u_3^2}{c^2} \right) \right]^{\frac{1}{2}} = c$$

The magnitude then tends to c for any admissible components u_2, u_3 . The velocity of this event in an inertial reference frame with relative motion \mathbf{v} is then $\lim_{u_1 \rightarrow c} \{\mathbf{v} \otimes \mathbf{u}\} = (v_1, v_2, v_3) \otimes \lim_{u_1 \rightarrow c} (u_1, u_2, u_3)$

$$\lim_{u_1 \rightarrow c} \{\mathbf{v} \otimes \mathbf{u}\} = \left(\lim_{u_1 \rightarrow c} \left\{ \frac{v_1 + u_1}{1 + \frac{v_1 u_1}{c^2}} \right\}, \frac{v_2 + u_2}{1 + \frac{v_2 u_2}{c^2}}, \frac{v_3 + u_3}{1 + \frac{v_3 u_3}{c^2}} \right) = \left(c, \frac{v_2 + u_2}{1 + \frac{v_2 u_2}{c^2}}, \frac{v_3 + u_3}{1 + \frac{v_3 u_3}{c^2}} \right).$$

The magnitude of the velocity in the primed frame $\mathbf{u}' = \mathbf{v} \otimes \mathbf{u}$ is then

$$\lim_{u_1 \rightarrow c} \|\mathbf{v} \otimes \mathbf{u}\|_d = \lim_{u_1 \rightarrow c} \left\{ c \left[1 - \prod_{j=1}^3 \left(1 - \left(\frac{v_j + u_j}{c(1 + v_j u_j / c^2)} \right)^2 \right) \right]^{\frac{1}{2}} \right\},$$

but the limit of the factor with $j = 1$ in the product is zero

$$\lim_{u_1 \rightarrow c} \left\{ 1 - \left(\frac{v_j + u_j}{c(1 + v_j u_j / c^2)} \right)^2 \right\} = 1 - \left(\frac{v_j + c}{c(1 + v_j / c)} \right)^2 = 0.$$

The magnitude thus approaches the velocity of light $\lim_{u_1 \rightarrow c} \|\mathbf{v} \otimes \mathbf{u}\|_d = c$ for all admissible velocities $\mathbf{v} = (v_1, v_2, v_3)$. Therefore, *the proposed framework is in accordance with the constancy of the speed of light limit regardless of the motion of the source.*

4. SEQUENTIAL ADDITION OF VELOCITIES

The transformation of velocities between inertial frames (5) can also be written in terms of parallel and perpendicular contributions [8, p.265], [9, p.523]

$$\mathbf{u}_{02\parallel} = \frac{\mathbf{v}_{12} + \mathbf{u}_{01\parallel}}{\left(1 + \frac{\mathbf{v}_{12} \cdot \mathbf{u}_{01}}{c^2} \right)}, \quad \mathbf{u}_{02\perp} = \frac{\mathbf{u}_{01\perp}}{\gamma_{v_{12}} \left(1 + \frac{\mathbf{v}_{12} \cdot \mathbf{u}_{01}}{c^2} \right)}, \quad (6)$$

where \mathbf{v}_{12} is the relative velocity of the frame \mathcal{F}_1 observed from the frame \mathcal{F}_2 , \mathbf{u}_{01} is the velocity of the event (\mathcal{F}_0 frame) in the \mathcal{F}_1 frame and \mathbf{u}_{02} is the velocity of the event observed from the \mathcal{F}_2 frame[†]. The parallel and perpendicular sub indices are the decomposition of the velocities $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ with respect to the relative velocity \mathbf{v} between frames such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}_{\parallel} \cdot \mathbf{v} = |\mathbf{u}_{\parallel}| |\mathbf{v}|$.

4.1 Cartesian decomposition

Consider an event to be at rest $\mathbf{u} = (0, 0, 0)$ in a reference frame, say the object frame \mathcal{F}_0 . Let the object move with velocity $\mathbf{v}_{01} = (v_{01x}, 0, 0)$ in the x direction[‡] relative to a frame \mathcal{F}_1 . Let the frame \mathcal{F}_1 in turn move with

[†]the common primed, unprimed notation used in the literature has not been employed here to avoid confusion between many-primed frames.

[‡]notation $v_{\#1 \#2 L}$, velocity of frame with first sub-index $\#1$ with respect to frame with second sub-index $\#2$, projection L (if applicable).
moving observation projection
frame frame direction

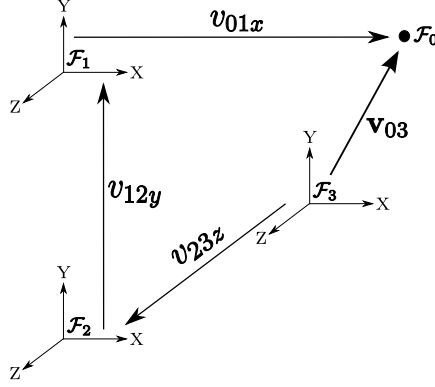


Figure 2. Sequence of three frames with orthogonal relative velocities. The object in frame \mathcal{F}_0 has relative velocity v_{01x} with respect to frame \mathcal{F}_1 . This frame, in turn, has relative velocity v_{12y} with respect to frame \mathcal{F}_2 , and frame \mathcal{F}_2 has relative velocity v_{23z} with respect to frame \mathcal{F}_3 . The velocity of the object frame \mathcal{F}_0 with respect to frame \mathcal{F}_3 in an arbitrary direction is \mathbf{v}_{03} given by (7) in SR. The velocity of the frame \mathcal{F}_0 with respect to \mathcal{F}_3 in the sequence invariant proposal is $\mathbf{v}_{03}^{(\text{SIR})} = (v_{01x}, v_{02y}, v_{03z})$. In either scheme the magnitude is given by (8).

velocity $\mathbf{v}_{12} = (0, v_{12y}, 0)$ in the y direction with respect to a frame \mathcal{F}_2 . The velocity of the object relative to the frame \mathcal{F}_2 is obtained from the SR composition of velocities

$$\mathbf{v}_{02} = (v_{02x}, v_{02y}, 0) = (v_{01x}\gamma_{12y}^{-1}, v_{12y}, 0) = (v_{01x}\gamma_{02y}^{-1}, v_{02y}, 0).$$

Let the frame \mathcal{F}_2 in turn move with velocity $\mathbf{v}_{23} = (0, 0, v_{23z}) = (0, 0, v_{03z})$ in the z direction with respect to a frame \mathcal{F}_3 . The velocity of the object \mathcal{F}_0 relative to the frame \mathcal{F}_3 in the SR composition of velocities is

$$\mathbf{v}_{03} = (v_{03x}, v_{03y}, v_{03z}) = (v_{01x}\gamma_{02y}^{-1}\gamma_{03z}^{-1}, v_{02y}\gamma_{03z}^{-1}, v_{03z}). \quad (7)$$

A Lorentz boost with arbitrary admissible velocity components $\mathbf{u} = (u_x, u_y, u_z)$ may be decomposed in a sequence of three successive orthogonal Lorentz boosts as shown in figure 2. However, such a decomposition is not unique since the order of the boosts may be interchanged, *i.e.* the other five possible permutations are

$$\begin{aligned} \mathbf{v}_{03} (v_y \rightarrow v_x \rightarrow v_z) &= (v_x\gamma_z^{-1}, v_y\gamma_x^{-1}\gamma_z^{-1}, v_z), \\ \mathbf{v}_{03} (v_x \rightarrow v_z \rightarrow v_y) &= (v_x\gamma_z^{-1}\gamma_y^{-1}, v_y, v_z\gamma_y^{-1}), \\ \mathbf{v}_{03} (v_z \rightarrow v_x \rightarrow v_y) &= (v_x\gamma_y^{-1}, v_y, v_z\gamma_x^{-1}\gamma_y^{-1}), \\ \mathbf{v}_{03} (v_z \rightarrow v_y \rightarrow v_x) &= (v_x, v_y\gamma_x^{-1}, v_z\gamma_x^{-1}\gamma_y^{-1}), \\ \mathbf{v}_{03} (v_y \rightarrow v_z \rightarrow v_x) &= (v_x, v_y\gamma_z^{-1}\gamma_x^{-1}, v_z\gamma_x^{-1}), \end{aligned}$$

where the arrows represent the composition sequence. Furthermore, these boost sequences have different orientations and the appropriate rotation should be considered in each case if an identical frame to the single boost is required.

The velocity $\mathbf{u} = \mathbf{v}_{03} (v_x \rightarrow v_y \rightarrow v_z)$ in terms of the sequence of three boosts (7)

$$\mathbf{v}_{03} = \left(v_x \left(1 - \frac{v_y^2}{c^2}\right)^{\frac{1}{2}} \left(1 - \frac{v_z^2}{c^2}\right)^{\frac{1}{2}}, v_y \left(1 - \frac{v_z^2}{c^2}\right)^{\frac{1}{2}}, v_z \right)$$

has a squared magnitude $|\mathbf{u}|^2 = u_x^2 + u_y^2 + u_z^2 = v_{03x}^2 + v_{03y}^2 + v_{03z}^2$ given by

$$|\mathbf{u}|^2 = |\mathbf{v}_{03}|^2 = v_x^2 \left(1 - \frac{v_y^2}{c^2}\right) \left(1 - \frac{v_z^2}{c^2}\right) + v_y^2 \left(1 - \frac{v_z^2}{c^2}\right) + v_z^2.$$

If the products are expanded, we obtain[§]

$$|\mathbf{v}_{03}|^2 = v_x^2 + v_y^2 + v_z^2 - \frac{v_x^2 v_y^2}{c^2} - \frac{v_y^2 v_z^2}{c^2} - \frac{v_z^2 v_x^2}{c^2} + \frac{v_x^2 v_y^2 v_z^2}{c^4},$$

that may be factored again into

$$|\mathbf{v}_{03}|^2 = c^2 \left[1 - \left(1 - \frac{v_x^2}{c^2} \right) \left(1 - \frac{v_y^2}{c^2} \right) \left(1 - \frac{v_z^2}{c^2} \right) \right]. \quad (8)$$

but this result is equal to the expression for the magnitude in the new proposal (2). In this scheme, the order of the boosts or whether a single boost is performed does not affect the magnitude of the velocity or its projections. Since the composition of velocities is identical regardless of the sequence or simultaneity in the present proposal, we refer to it as *sequentially invariant relativity* (SIR). The velocity components in the new scheme corresponding to the above magnitude are $\mathbf{v}_{03}^{(\text{SIR})} = (v_x, v_y, v_z)$. In contrast, as it is well known, the order of the sequence in SR leaves the magnitude of the velocity invariant but its projections differ. If the boosts are simultaneous, or from another point of view, relative to the same observer frame, the magnitude of the velocity in SR is $\sqrt{v_x^2 + v_y^2 + v_z^2}$ and is no longer equal to the sequential application of the three velocities.

The magnitude of the velocity vector in special relativity (SR) for a sequence of three orthogonal Lorentz boosts, say $v_x \rightarrow v_y \rightarrow v_z$, is equal to the magnitude in the new proposal (SIR) of a velocity vector with projections (v_x, v_y, v_z) applied either sequentially or simultaneously.

It may be useful to rewrite the velocity transformations in the new sequentially invariant relativity proposal in terms of parallel and perpendicular contributions as commonly stated in special relativity (6). From the velocity composition rule in the present proposal (1),

$$\mathbf{u}_{02\parallel} = \frac{\mathbf{v}_{12} + \mathbf{u}_{01\parallel}}{\left(1 + \frac{\mathbf{v}_{12} \cdot \mathbf{u}_{01\parallel}}{c^2} \right)}, \quad \mathbf{u}_{02\perp} = \mathbf{u}_{01\perp}, \quad (9)$$

that is, the parallel velocity transforms in the same fashion as in SR whereas the perpendicular velocity in SIR (in contrast with SR) remains the same regardless of the parallel velocity contribution.

5. COMPOSITE-VELOCITY RECIPROCITY PRINCIPLE

The reciprocity principle regarding velocities may be stated as follows: Let the velocity of an inertial frame \mathcal{F}_{obj} relative to another inertial reference frame \mathcal{F}_{lab} be \mathbf{u}' , then reciprocally, the velocity of \mathcal{F}_{lab} relative to \mathcal{F}_{obj} is $-\mathbf{u}'$. A rigorous treatment of the reciprocity principle has been derived from the postulates of (i) homogeneity of space-time, (ii) isotropy of space and (iii) the equivalence of inertial frames.¹⁰

Allow for the velocity \mathbf{u}' to be composed by the addition of two velocities as shown in figure 3. On the one hand, the composite velocity seen from the laboratory reference frame \mathcal{F}_{lab} is the resultant of the velocity \mathbf{u} of the frame \mathcal{F}_{obj} with respect to a vehicle frame \mathcal{F}_{veh} and the velocity \mathbf{v} of this vehicle frame with respect to the laboratory frame \mathcal{F}_{lab} . The addition of composite velocities is $\mathbf{u}'_{lab} = \mathbf{v} \boxplus \mathbf{u}$ where the squared plus symbol stands for whatever addition operation is defined. On the other hand, the composite velocity as seen from the object reference frame \mathcal{F}_{obj} consists of a velocity $-\mathbf{v}$ of the frame \mathcal{F}_{lab} with respect to \mathcal{F}_{veh} and a velocity $-\mathbf{u}$ of \mathcal{F}_{veh} with respect to \mathcal{F}_{obj} . Addition of composite velocities then reads $\mathbf{u}'_{obj} = (-\mathbf{u}) \boxplus (-\mathbf{v}) = -(\mathbf{u} \boxplus \mathbf{v})$. Due to the reciprocity principle, $\mathbf{u}'_{lab} = -\mathbf{u}'_{obj}$ and therefore the commutative equality $\mathbf{v} \boxplus \mathbf{u} = \mathbf{u} \boxplus \mathbf{v}$ should be fulfilled.

However, the addition of velocities in special relativity is neither commutative nor associative. The lack of commutation leads to an apparent inconsistency since the reciprocity principle requires the addition of velocities to be commutative. However, if the coupling between relative velocities and frame orientations is considered, the velocity reciprocity may be recovered.¹¹ The introduction of a rotation operator in the velocity addition laws, has allowed for gyro-commutative and gyro-associative laws to produce a gyrogroup.⁶

[§]this magnitude is the same for any of the boost sequences.

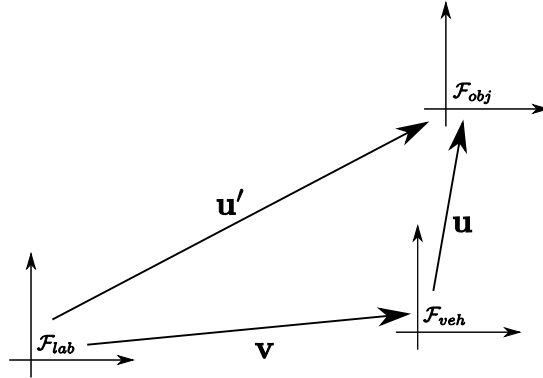


Figure 3. Velocity reciprocity principle - A laboratory frame \mathcal{F}_{lab} observes a vehicle \mathcal{F}_{veh} with velocity \mathbf{v} , the vehicle emits an object \mathcal{F}_{obj} with velocity \mathbf{u} , the object is seen with velocity \mathbf{u}' from the laboratory frame. The object, should reciprocally observe the laboratory frame with velocity $-\mathbf{u}'$.

In contrast, the composition operation definition (1) that has been introduced here is both commutative and associative. In order to ascertain the reciprocity principle in the new schema, multiplication by a scalar has to be defined: Multiplication of a velocity element by a real scalar in SIR follows the distributivity of scalar multiplication over vector addition in a linear space $\lambda \mathbf{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$. In particular, the inverse additive element of an arbitrary element $\mathbf{u} = (u_1, u_2, u_3)$ may be written as $-\mathbf{u} = (-u_1, -u_2, -u_3) = (-1)(u_1, u_2, u_3)$.

In SIR, the composition of velocities in the laboratory frame is $\mathbf{u}'_{lab} = \mathbf{v} \otimes \mathbf{u}$. On the other hand, the composition of velocities from the object frame is $\mathbf{u}'_{obj} = (-\mathbf{u}) \otimes (-\mathbf{v})$. Thus no paradox arises since it follows immediately from the composition of velocities (1) that the reciprocity principle is fulfilled because $\mathbf{u}'_{obj} = (-\mathbf{u}) \otimes (-\mathbf{v}) = -(\mathbf{u} \otimes \mathbf{v}) = -\mathbf{u}'_{lab}$. Velocity is then a purely relative quantity with no preferred state of motion.

6. THOMAS ROTATION

In the preceding two sections the rotation of frames cropped up due to the application of successive boosts and in order to fulfill the velocity reciprocity principle. Recall that the composition of two successive Lorentz boosts is equivalent to a single boost followed or preceded by a rotation thus giving rise to the Thomas gyration. The rotational effect is even clearer if three successive Lorentz boosts with relative velocities adding up to zero are considered; The initial and final states, say both at rest, are then in general, nonetheless rotated.¹² The infinitesimal rotation is often referred to as the Thomas precession. The Thomas precession most celebrated past success is that it accounts for a factor of two in the derivation of the electron spin [13, pp. 106-119]. Its present success lies in the explanation of various paradoxes such as the Mocanu paradox that are ultimately related to the fulfillment of the reciprocity principle.

The difficulties for grasping the Thomas precession have given rise to many communications.¹⁴ These difficulties are not unfounded for there is no fundamental reason, a priori, to request that two Lorentz boosts should involve a rotation. Furthermore, the (Galilean) addition of velocities of our everyday experience is commutative and associative even in three dimensions and does not exhibit any sort of rotation. It should be recalled that this effect was put forward more than twenty years after the special theory was presented and even then it was a great surprise to many of the founders of the theory.^{13, 14} It is indeed a consequence of the addition of velocities rule as stated in Einstein's proposal, albeit not necessarily a desirable one. What is imperative in the velocity transformations is that the velocity of light is not surpassed. This crucial requirement is equally fulfilled by the present proposal as we have already shown.

The lack of commutativity and associativity in the SR addition of velocities together with an invariant magnitude for the different sequences is the mathematical structure responsible for rotations. The commutativity and associativity SR addition laws in one dimension¹⁵ have to be carefully generalized to three dimensions where SR laws are neither commutative nor associative.¹⁶ Much as it is counter intuitive to recognize the Thomas rotation, once comprehended, it becomes part of our understanding of special relativity theory. Eighty years have elapsed since these ideas were first put forward and albeit slowly, they have become now days part of the

hard core of the theory, specially since the introduction of gyro-groups and their geometrical interpretation by Ungar and other workers.^{7,17}

It is not manifest whether the present approach is involving intrinsically a rotation operator in a comparable fashion as the explicit introduction of a Thomas rotation operator in gyro-groups.⁷ Alternatively, it is possible that this scheme is not involving a rotation of the inertial frames at all. A definitive answer to these questions requires a detailed analysis of the time and space transformations consistent with the present velocity addition rules. Nonetheless, the commutativity and associativity of the composition of velocities presented here is indicative that no rotations are taking place since rotations in three dimensions differ depending on their order (commutativity). A sequence of three or four boosts in SR can be devised such that the initial and final frames are at rest with respect to each other.¹⁴ The sequence $v\hat{e}_x \rightarrow \frac{v}{\sqrt{1+\beta^2}}\hat{e}_y \rightarrow \frac{-v}{\sqrt{1+\beta^2}}\hat{e}_x \rightarrow -v\hat{e}_y$ with $\beta = v/c$ returns the system to rest with its axes rotated in the x - y plane by $\theta = \arcsin(\beta^2)$. In sharp contrast, the above sequence does not take the system back to rest with respect to the initial frame in SIR. The previous sequence of four boosts for an event initially at rest give a final frame with velocity

$$(0, 0, 0) \rightarrow \left(\frac{v(-\sqrt{\beta^2+1}+1)}{\beta^2-\sqrt{\beta^2+1}}, \frac{v(\sqrt{\beta^2+1}-1)}{\beta^2-\sqrt{\beta^2+1}}, 0 \right).$$

However, in the present proposal we can use the sequence $v_x\hat{e}_x \rightarrow v_y\hat{e}_y \rightarrow -v_x\hat{e}_x \rightarrow -v_y\hat{e}_y$ for an arbitrary event initially with velocity (u_x, u_y, u_z) , the composition of velocities is then

$$\left(\frac{\frac{u_x+v_x}{1+u_x v_x} - v_x}{1 - \frac{v_x(u_x+v_x)}{1+u_x v_x}}, \frac{\frac{u_y+v_y}{1+u_y v_y} - v_y}{1 - \frac{v_y(u_y+v_y)}{1+u_y v_y}}, u_z \right)$$

that can be simplified to the initial velocity (u_x, u_y, u_z) . This result is akin to our non relativistic experience where the sequential addition the velocities in different directions does not produce a rotation.

If the present proposal does not involve rotations for non-collinear velocity compositions, it is then crucial to see if we can live without the Thomas precession. The first hurdle is to have c bounded velocity transformations that do not involve rotations, an issue that is possible as we have shown here. The second immediate hurdle is to explain the electron spin without incurring into the factor of two inconsistency that led Thomas to include the frame precession as the particle changes direction.

7. DEFORMED SPECIAL RELATIVITY SCHEME

The present proposal may be formulated along the lines of deformed Minkowski metric schemes. These deformations can be dependent for example on the type of interaction,⁵ Finslerian generalizations of Riemannian geometry¹⁸ or the space anisotropy. The general deformed velocity magnitude in the present scheme is

$$\|\mathbf{u}\|_{def}^2 = \frac{b_1^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}u_1^2 + \frac{b_2^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}u_2^2 + \frac{b_3^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}u_3^2 - \frac{b_{12}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})} \frac{u_1^2 u_2^2}{c^2} - \frac{b_{23}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})} \frac{u_2^2 u_3^2}{c^2} - \frac{b_{31}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})} \frac{u_3^2 u_1^2}{c^2} + \frac{b_{123}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})} \frac{u_1^2 u_2^2 u_3^2}{c^4}, \quad (10)$$

where the metric coefficients $b_j^2(\{\mathcal{O}\})$ are real positive functions. The set $\{\mathcal{O}\}$ represents a set of non-metric observable variables. An energy dependent deformation $\{\mathcal{O}\} \rightarrow \mathcal{E}$ has been chosen in several frameworks. The isotropic version of the above deformed velocity magnitude is

$$\|\mathbf{u}\|_{def}^2 = \frac{b_{s1}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}(u_1^2 + u_2^2 + u_3^2) - \frac{1}{c^2} \frac{b_{s2}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}(u_1^2 u_2^2 + u_2^2 u_3^2 + u_3^2 u_1^2) + \frac{1}{c^4} \frac{b_{123}^2(\{\mathcal{O}\})}{b_0^2(\{\mathcal{O}\})}u_1^2 u_2^2 u_3^2. \quad (11)$$

Table 1. Metric coefficients in different deformation limits.

SIR limit	DSR limit	SR limit
$b_0^2 \rightarrow 1, b_{s_1}^2 \rightarrow 1$ $b_{s_2}^2 \rightarrow 1, b_{123}^2 \rightarrow 1$	$b_{s_2}^2 \rightarrow 0, b_{123}^2 \rightarrow 0$	$b_0^2 \rightarrow 1, b_{s_1}^2 \rightarrow 1$ $b_{s_2}^2 \rightarrow 0, b_{123}^2 \rightarrow 0$
composition is independent of sequence	energy dependent metric is allowed	composition is sequence dependent

If the metric coefficients are equal to one $b_0^2(\{\mathcal{O}\}) = b_{s_1}^2(\{\mathcal{O}\}) = b_{s_2}^2(\{\mathcal{O}\}) = b_{123}^2(\{\mathcal{O}\}) = 1$, the SIR (Sequentially invariant relativity) velocity magnitude given by (4) is recovered. On the other hand, if the coefficients involving two or more velocity products are zero $b_{s_2}^2(\{\mathcal{O}\}) = b_{123}^2(\{\mathcal{O}\}) = 0$, the DSR version proposed by Cardone and Mignani⁵ is obtained. If, in addition $b_0^2(\{\mathcal{O}\}) = b_{s_1}^2(\{\mathcal{O}\})$, the SR Minkowski's three component vector metric is recovered. The metric proposal with deformed metric coefficients (11) allows for the possibility of a smooth space-time deformation. The velocity transformations may then exhibit a sequentially invariant mapping in the SIR limit and a sequence dependent mapping in the SR limit.

8. FINAL REMARKS

The composition theorem for velocities together with the magnitude presented here have been proved to be consistent with the two postulates of special relativity, namely the invariance of the speed of light (speed of light as a limiting value for inertial frames) and the relativity of inertial frames (no preferred reference frame). This composition operation is commutative and associative in sharp contrast with Einstein's addition of velocities obtained from the Lorentz transformations that is neither commutative nor associative in two or three spatial dimensions.

Regarding similarities, the velocity composition law proposed here is identical to the special relativity (SR) velocity addition rule in one dimension. In the low velocity limit, the proposed composition operation and its magnitude approach the Galilean addition of velocities and the Euclidean vector magnitude.

However, major differences arise when non parallel velocities are considered. The present proposal, due to its group properties, is independent of the sequence in which the frames with different velocities are composed. For this reason, we refer to the present framework as sequentially invariant relativity (SIR). This framework does not involve a Thomas rotation for a sequence of non collinear velocity transformations. On the other hand, the magnitude of the velocity in this proposal involves the product of orthogonal velocity components. The nature of these nonlinear terms needs to be elucidated. However, a curious result is that the SR magnitude for a sequential application of boosts in orthogonal directions is equal to the magnitude given in SIR.

Observational data should be in accordance with the predictions of the mathematical structure and its physical interpretation as mentioned in the first lines of this manuscript. This agreement is often quoted as a test of the validity of the physical model. From this point of view, it is interesting to recall recent observations of fast receding astronomical objects, which exhibit very high transverse velocities leading to apparently superluminal speeds.^{19,20} It should be most interesting to examine the observed data within the present formulation. Subluminal speeds are to be expected since the proposed scheme always leads to subluminal velocities even if the parallel and transverse velocities are arbitrarily close to c .

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