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Analytic approximation to the harmonic oscillator equation with a sub-period time dependent parameter☆

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Abstract

Analytic solutions are presented for the time dependent harmonic oscillator equation as well as for the corresponding amplitude (Ermakov) and phase equations. The solutions are adequate approximations for an initially constant amplitude and a time dependent parameter that varies continuously in a time much shorter than the characteristic period. The amplitude and phase representation of the solution is shown not to be unique. The relationship between the phase independent amplitudes is derived from the orthogonal functions invariant.

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1. Introduction

The classical as well as the quantum time dependent harmonic oscillator equations have received renewed interest due to the development of exact invariants [1]. In the former case, analytical solutions to this equation have been presented for some time dependent parameter functions [2]. In the latter case, analytical solutions have been obtained for the one-dimensional time dependent Schrödinger equation [3]. These solutions are expressed in terms of a c-number function that obeys either the time dependent harmonic oscillator (TDHO) equation or an auxiliary Ermakov equation [4]. In either case, it is of interest to describe the solutions in terms of analytic results valid throughout the motion of the system.

In the slowly varying time parameter regime, solutions fulfilling adiabatic invariance have been successfully used since the development of statistical mechanics [5]. In contrast, problems involving time dependent parameters of the order of the fundamental oscillation of the system or faster need to be tackled using different techniques such as exact invariants. The sub-period regime is defined here when the time dependent parameter varies appreciably in

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an interval much shorter than the fundamental period of the system although not necessarily infinitesimal. A step function that varies abruptly in time is the fast limit of the sub-period regime. The relevance of this problem in many areas of physics has been pointed out before [6]. The step function limit is usually tackled through piecewise integration of the appropriate equations in different intervals. The solutions at each interval are then brought together via continuity conditions at the interfaces. For example, a linear sweep time dependence has been solved using this procedure [7]. However, the derivative of the time dependent parameter in this approach is not continuous at the interfaces.

The amplitude and phase representation of the coordinate variable has proved very useful for tackling time dependent problems [8]. The linear TDHO equation for the coordinate variable $x(t)$ yields a nonlinear equation for the amplitude $g(t)$. The latter equation is numerically tractable and appropriate boundary conditions may be imposed to produce smooth amplitude and phase functions [9]. Furthermore, the linear superposition principle of independent solutions in the coordinate variable has been translated into a nonlinear superposition relationship of these variables for the amplitude function [10]. The TDHO equation subject to pulsed parametric excitation of the time dependent parameter has been recently reported using the amplitude approach [11]. The authors' attention is focused on the relationship between pulse parameters and transitions. They exhibit analytic expressions for the asymptotic behavior of the harmonic oscillator that will be compared with the appropriate limits of the results presented here.

The present communication describes an approximate analytic solution to the classical TDHO equation and its corresponding Ermakov pair for a time dependent parameter that varies in a sub-period interval. This function is requested to vary monotonically in a time span much smaller than the fundamental period of the system from one steady value to another but its specific choice is otherwise left free. The boundary condition imposed throughout is that the asymptotic initial amplitude is constant. The proposed solution, which holds for all times, is written in terms of two complex opposite phase functions with different amplitudes and a constant phase shift. The orthogonal functions invariant together with a continuity condition are then invoked in order to establish a relationship between the amplitudes and the phase involved. The Ermakov equation for the amplitude is evaluated for a hyperbolic tangent function and the analytic and numerical results are compared. The solutions corresponding to the frequency equation are also presented. The time dependent amplitude and nonlinear frequency solutions are thereafter translated into a phase independent amplitude and a linear frequency. These two representations of the proposed solution are finally assessed for some specific cases.

2. Form of the proposed solution

The starting point is the TDHO equation:

$$\ddot{x}(t) + \Omega^2(t)x(t) = 0, \tag{1}$$

where $x(t)$ is the harmonic variable and $\Omega^2(t)$ a continuous real parameter. Let the solution $x(t)$ be expressed in terms of a complex function with amplitude $g(t)$ and phase $\gamma(t)$:

$$x(t) = g(t)\,\mathrm{e}^{\mathrm{i}\gamma(t)}. \tag{2}$$

The amplitude function then obeys the Ermakov equation:

$$\ddot{g}(t) + \Omega^2(t)g(t) - \frac{Q^2}{g(t)^3} = 0, \tag{3}$$

where Q is a constant of motion. The frequency, defined as the derivative of the phase function $\dot{\gamma}(t) \equiv \omega(t)$, obeys the equation:

$$\omega(t)\ddot{\omega}(t) - \tfrac{3}{2}\dot{\omega}^2(t) + 2[\omega^2(t) - \Omega^2(t)]\omega^2(t) = 0. \tag{4}$$

Consider the solution of the TDHO equation (1) to stem from the sum of two functions with opposite phase:

$$x(t) = g(t)\,\mathrm{e}^{\mathrm{i}\gamma(t)} = a(t)\,\mathrm{e}^{\mathrm{i}s(t)+\mathrm{i}\theta} + b(t)\,\mathrm{e}^{-\mathrm{i}s(t)+\mathrm{i}\theta}, \tag{5}$$

where θ is a constant phase shift. Notice that each of these functions on their own are not necessarily solutions to the TDHO equation due to the different time dependence of their amplitudes. Let us elucidate this statement. Allow for the two additive terms in the above equation to be written as $A(t)$ and $B(t)$; if $x(t) = A(t) + B(t)$ is a solution to the TDHO equation, then $\ddot{A}(t) + \Omega^2(t)A(t) = f(t)$ and $\ddot{B}(t) + \Omega^2(t)A(t) = -f(t)$. Since in general $f(t) \neq 0$, the solution (5) does not follow from a superposition principle. However, in the particular case when $f(t) = 0$, this solution indeed obeys a superposition relationship. The solution expressed in terms of a single exponential yields an amplitude:

$$g(t) = \sqrt{a^2(t) + b^2(t) + 2a(t)b(t)\cos(2s(t))} \tag{6}$$

and a phase

$$\gamma(t) = \arctan\left[\frac{a(t) - b(t)}{a(t) + b(t)}\tan(s(t))\right] + \theta. \tag{7}$$

The advantage of writing the solution in this way is that if we now consider a region where the time varying parameter is constant, then the amplitude functions $a(t), b(t)$ are constant while the phase $s(t)$ is a linear function of time. This assertion follows from the general solution to the TDHO equation (1) for constant parameter Ω_c, since then $x(t) = a\,\mathrm{e}^{\mathrm{i}s(t)+\mathrm{i}\theta} + b\,\mathrm{e}^{-\mathrm{i}s(t)+\mathrm{i}\theta}$, where a, b are constants and $s(t) = \Omega_\mathrm{c}t$. It is only in this particular circumstance that the forms (6) and (7) follow from the nonlinear superposition principle.

3. Invariant

The orthogonal functions invariant is an exact constant of motion given by [8,12]:

$$Q = x(t)\dot{x}_\perp(t) - x_\perp(t)\dot{x}(t), \tag{8}$$

where x and $x_\perp$ are orthogonal solutions to the TDHO equation. In the amplitude and phase representation this invariant reads

$$Q = g^2(t)\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}, \tag{9}$$

which in terms of the opposite phase functions variables is

$$Q = [a^2(t) - b^2(t)]\dot{s}(t) + [b(t)\dot{a}(t) - a(t)\dot{b}(t)]\sin(2s(t)). \tag{10}$$

In a constant parameter interval, the invariant is $Q = (a^2 - b^2)\dot{s}$, where a, b and $\dot{s}$ are time independent. Nonetheless, it is also approximately true even if the amplitudes are time dependent:

$$Q \approx [a^2(t) - b^2(t)]\dot{s}(t), \tag{11}$$

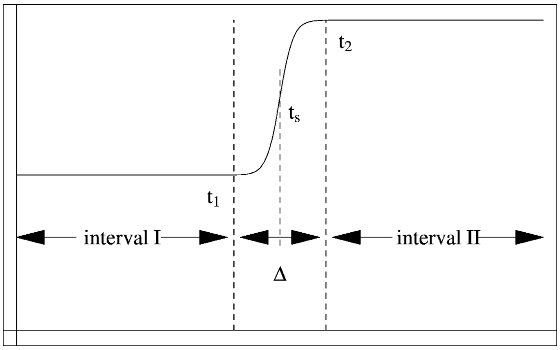

Fig. 1. Time dependent parameter function $\Omega(t)$ vs. t.

provided that

$$\left|2a^2(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{b(t)}{a(t)}\right)s(t)\right| \ll |Q|. \tag{12}$$

This inequality is fulfilled if the phase $s(t)$ is much smaller than the time derivative of the amplitudes ratio. It is of course also valid if the amplitude ratio has a very slow variation but we shall waive this limit since it leads to the well-known adiabatic regime.

Allow for the time dependent parameter $\Omega(t)$ to be a C_2 class function that evolves from a constant value Ω_1 within an interval I, to another constant value Ω_2 within the interval II as shown in Fig. 1. Consider an initial condition at a time t_1 within the interval I where the amplitude is constant $g(t) = a_1$, that is, the amplitudes of the opposite phase functions are $a(t) = a_1$ and $b(t) = 0$. This boundary condition will be maintained all the way through this work. The invariant is then $Q = a_1^2\Omega_1$. Let the parameter $\Omega(t)$ vary monotonically in a subsequent interval Δ centered at a time t_s and eventually reach the interval II where although the amplitude $g(t)$ is time dependent, the amplitudes $a(t)$, $b(t)$ are again constants, say a_2, b_2. The invariant in this interval is $Q = (a_2^2 - b_2^2)\Omega_2$. Since the invariant is a constant of motion, then

$$a_1^2\Omega_1 = (a_2^2 - b_2^2)\Omega_2. \tag{13}$$

3.1. Continuity condition

In the vicinity of the interval where the time dependent parameter has an appreciable variation, the invariant condition holds if the phase $s(t)$ is small within the interval. The solution in this region is then

$$\lim_{s(t)\ll 1}\{x(t)\} = [a(t) + b(t)]\,\mathrm{e}^{\mathrm{i}\theta}. \tag{14}$$

Allow for the coordinate variable $x(t)$ to be continuous since the position of an event may not change abruptly without violating fundamental physical laws. The relationship

$$a_1 = a(t) + b(t) \tag{15}$$

is then valid in the region Δ where the phase is small. The relation between the opposite phase amplitudes is then obtained from the invariant condition (11) and the above relationship:

$$a(t) = \frac{1}{2}a_1\left(1 + \frac{Q}{a_1^2\dot{s}(t)}\right), \qquad b(t) = \frac{1}{2}a_1\left(1 - \frac{Q}{a_1^2\dot{s}(t)}\right). \tag{16}$$

Since the phase is defined as the integral of the frequency and the frequency equation for constant Ω_{c} has a solution $\omega = \Omega_{\mathrm{c}}$, then the condition $s(t) \ll 1$ is fulfilled by allowing the integration to be centered at t_s. The phase is then proposed to be

$$s(t) = \int_{t_s}^{t} \Omega(t')\,\mathrm{d}t' \tag{17}$$

in the entire interval. This proposal may at first sight look like a slowly varying parameter approximation. However, the actual phase solution as shall be seen in Section 5 is given by (28). In the interval Δ, where the parameter has a strong time dependence, the phase $s(t)$ is not even an approximate solution to the phase equation (4). The crucial feature of this definition is that the lower integral limit has been chosen so that this function is vanishingly small in the vicinity of t_s. The time dependent amplitudes of the opposite phase functions may thus be written as

$$a(t) = \frac{1}{2} a_1 \left(1 + \frac{\Omega_1}{\Omega(t)}\right), \qquad b(t) = \frac{1}{2} a_1 \left(1 - \frac{\Omega_1}{\Omega(t)}\right). \tag{18}$$

The restriction (12) imposed on the time varying parameter:

$$\frac{\dot{\Omega}(t)}{\Omega^2(t)} \int_{t_s}^{t} \Omega(t')\,\mathrm{d}t' \ll 1, \tag{19}$$

reflects the fact that the phase should be small during the time where the parameter has a sharp variation.

4. Amplitude equation

The proposed solution for the amplitude function is then

$$g(t) = \frac{a_1}{\sqrt{2}} \sqrt{1 + \frac{\Omega_1^2}{\Omega^2(t)} + \left(1 - \frac{\Omega_1^2}{\Omega^2(t)}\right) \cos\left(2 \int_{t_s}^{t} \Omega(t')\,\mathrm{d}t'\right)}. \tag{20}$$

If this solution is inserted in the Ermakov equation (3), one obtains

$$\frac{\begin{array}{c} -[(\Omega_1^2 \sin^2 s + \Omega^2 \cos^2 s)\Omega\Omega_1^2 \sin 2s]\ddot{\Omega} + [(2\Omega_1^2 \sin^2 s + 3\Omega^2 \cos^2 s)\Omega_1^2 \sin^2 s]\dot{\Omega}^2 \\ -[(\Omega^4 \cos^3 s + 3\Omega^2\Omega_1^2 \cos s + \Omega_1^4 \cos s \sin^2 s)\Omega^2 \sin s]\dot{\Omega} \end{array}}{\Omega^6 \Omega_1^2 (\cos^2 s + (\Omega_1^2/\Omega^2) \sin^2 s)^{3/2}} \overset{?}{=} 0, \tag{21}$$

where s is given by (17). From this result it is clear that if Ω is constant, the numerator is zero since all terms depend on derivatives of Ω, thus fulfilling the differential equation. The solution (20) in this limit has the same form given by Thylwe and Korsch in the asymptotic harmonic oscillator form [11]. In the vicinity of the step, the argument of the trigonometric functions is small and the equation reads

$$-\frac{1}{\Omega^3 \Omega_1^2} s^2 \ddot{\Omega} + 3\left(\frac{s\dot{\Omega}}{\Omega^2}\right)^2 - \left(\frac{\Omega^2}{\Omega_1^2} + 3\right)\frac{s\dot{\Omega}}{\Omega^2} \overset{?}{=} 0 \tag{22}$$

but due to the restriction (19), all terms are less than unity, thus providing an adequate first order approximation to the differential equation. Another way to look at the validity of the solution in the region where the opposite phase amplitudes are time dependent is the following: since the phase is small in this region, the cosine function in the amplitude (20) may be approximated to 1 and the amplitude is then time independent thus fulfilling the differential equation.

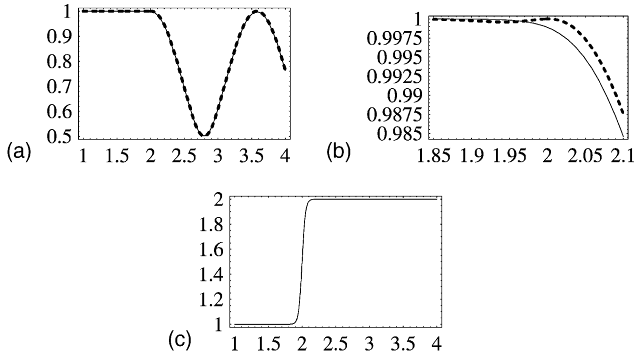

Fig. 2. (a) Amplitude $g(t)$ differential equation numerical solution (solid line) and analytic approximation (dashed line) for the time dependent parameter $\Omega(t)$ shown in (c) with $\beta = 1$, $\alpha = 20$ and $t_s = 2$; the initial conditions are $g(t = -\infty) = 1$, $\dot{g}(t = -\infty) = 0$. (b) Detail of the numerical and analytic solutions in the region where the parameter varies rapidly.

In order to present a comparison between the analytical solution proposed above and the numerical solution obtained from the amplitude differential equation (3), let the step function be represented by a hyperbolic tangent:

$$\Omega(t) = \omega_0[1 + \tfrac{1}{2}\beta(1 + \tanh[\alpha(t - t_s)])], \tag{23}$$

where β is the height of the step and α a measure of the slope at t_s. The variation within 90% of its initial and final values is given by $\Delta \simeq 2.2/\alpha$ [13]. The step function limit is obtained for $\alpha \to \infty$. Fig. 2 shows the numerical solution of the Ermakov equation (3) as well as the analytical approximation given by Eq. (20). The coincidence of these two curves is remarkably good even in the region where the parameter has a sharp variation. For the numerical calculations, $\alpha > 20$ yields a deviation between curves of less than 0.2% as shown in the detail of Fig. 2. This value corresponds to a time dependent parameter variation in 1/10 or less of the characteristic period.

5. Phase equation

The general solution for the phase for constant Ω_c is

$$\gamma = \arctan\left[\frac{a-b}{a+b}\tan s(t)\right] + \theta, \tag{24}$$

where $s(t) = \int \Omega_c \, dt$. Therefore the frequency equation (4), in addition to $\omega = \Omega_c$ also admits a second solution:

$$\omega = \dot{\gamma} = \frac{((a-b)/(a+b))\dot{s}(t)}{((a-b)/(a+b))^2 \sin^2(s(t)) + \cos^2(s(t))}. \tag{25}$$

For the time dependent function $\Omega(t)$, we substitute the amplitude (18) and phase (17) proposals to obtain

$$\omega(t) = \dot{\gamma} = \frac{\Omega_1}{(\Omega_1/\Omega(t))^2 \sin^2\left(\int_{t_s}^{t} \Omega(t')\,dt'\right) + \cos^2\left(\int_{t_s}^{t} \Omega(t')\,dt'\right)}. \tag{26}$$

This approximate analytical solution for the frequency together with the numerical solution to Eq. (4) are plotted in Fig. 3. The difference between the two curves is less than half percent for $\alpha = 20$, this deviation becomes much smaller for larger α. Notice that the frequency remains time dependent even after the parameter becomes constant again.

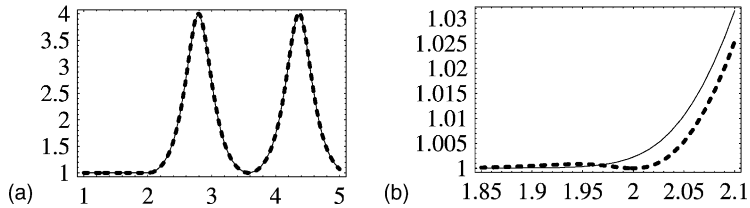

Fig. 3. (a) Frequency $\omega(t)$ differential equation numerical solution (solid line) and analytic approximation (dashed line) for the same time dependent parameter and initial conditions specified in Fig. 2c; (b) close up of the results in the vicinity of the parameter rapid variation.

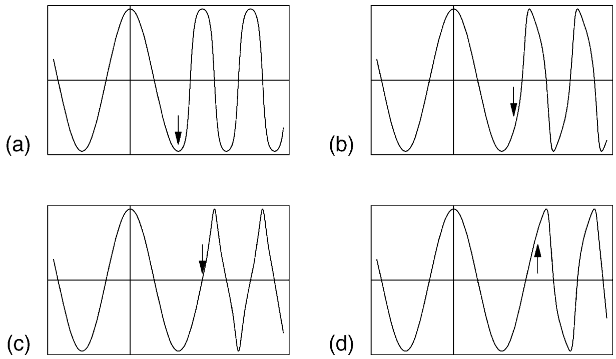

Fig. 4. Plots depicting the dependence of $\cos\gamma(t)$ vs. t; the arrow shows the time t_s where parameter sudden change is centered: (a) π, (b) $5\pi/4$, (c) $3\pi/2$, (d) $7\pi/4$.

The proposed solution for the phase is

$$\gamma(t) = \arctan\left[\frac{\Omega_1}{\Omega(t)}\tan\int_{t_s}^{t}\Omega(t')\,\mathrm{d}t'\right] + \theta. \tag{27}$$

The constant θ is obtained from the evaluation of the phase initial condition. In interval I, $\gamma_1(t) = \Omega_1(t - t_s) + \theta$; let $\gamma_1(t_0) = 0$, thus $\theta = (t_s - t_0)\Omega_1$ and the phase is

$$\gamma(t) = \arctan\left[\frac{\Omega_1}{\Omega(t)}\tan\int_{t_s}^{t}\Omega(t')\,\mathrm{d}t'\right] + (t_s - t_0)\Omega_1. \tag{28}$$

Although the phase is linear in interval I, it becomes a nonlinear function of time after the time dependent parameter abrupt change. The cosine of the phase is plotted in Fig. 4 to illustrate the skew behavior of these curves. Their shapes depend on the specific time t_s where the parameter varies sharply. The phase solution (28) in the asymptotic limit of constant Ω is equal, mutatis mutandis, with the nonlinear phase obtained in a previous report (their Eq. (23)) for the asymptotic form [11]. Such a form is in fact a consequence of the nonlinear superposition relation [10].

6. Phase independent amplitude and frequency

The TDHO solution $x(t)$ has been expressed in terms of the amplitude $g(t)$—that is dependent on the phase $s(t)$—and the nonlinear phase $\gamma(t)$. However, it may also be written in terms of a phase $s(t)$ independent amplitude $G(t)$ and a linear phase $\Gamma(t)$. To this end, recall the solution (5) in complex additive notation:

$$x = [a\cos(s(t)+\theta) + b\cos(s(t)-\theta)] + \mathrm{i}[a\sin(s(t)+\theta) - b\sin(s(t)-\theta)]. \tag{29}$$

The real part is unaltered if we interchange the variables s and θ:

$$x^{(\mathrm{inv})} = [a\cos(s(t)+\theta) + b\cos(s(t)-\theta)] + \mathrm{i}[a\sin(s(t)+\theta) + b\sin(s(t)-\theta)], \tag{30}$$

thus $\mathbb{R}e\{x\} = \mathbb{R}e\{x^{(\mathrm{inv})}\}$. Returning to the single exponential amplitude and phase notation:

$$\mathbb{R}e\{x^{(\mathrm{inv})}\} = \sqrt{a^2+b^2+2ab\cos(2\theta)}\cos\left(\arctan\left[\frac{a-b}{a+b}\tan\theta\right] + s(t)\right). \tag{31}$$

However, the imaginary part becomes

$$\mathbb{I}m\{x^{(\mathrm{inv})}\} = \sqrt{a^2+b^2-2ab\cos(2\theta)}\sin\left(\arctan\left[\frac{a+b}{a-b}\tan\theta\right] + s(t)\right) \tag{32}$$

because $b \to -b$ in addition to interchanging variables, thus $\mathbb{I}m\{x\} \neq \mathbb{I}m\{x^{(\mathrm{inv})}\}$.

The phase dependent amplitude $g(t)$ is then mapped into an $s(t)$ independent amplitude:

$$g(t) \to G(t) = \sqrt{a^2(t)+b^2(t)+2a(t)b(t)\cos(2\theta)} \tag{33}$$

and for a sub-period function with opposite phase amplitudes given by (18):

$$G(t) = \frac{a_1}{\sqrt{2}}\sqrt{1+\frac{\Omega_1^2}{\Omega^2(t)} + \left(1-\frac{\Omega_1^2}{\Omega^2(t)}\right)\cos(2(t_s-t_0)\Omega_1)}. \tag{34}$$

The amplitude $G(t)$ is then time independent even for constant $\Omega \neq \Omega_1$. It is interesting to notice that although the phase dependent amplitude $g(t)$ is continuous, the amplitude $G(t)$ is not continuous in the limit of an abrupt step.

The nonlinear phase $\gamma(t)$ is correspondingly mapped into a linear $s(t)$ dependent phase:

$$\gamma(t) \to \Gamma(t) = s(t) + \arctan\left[\frac{a(t)-b(t)}{a(t)+b(t)}\tan\theta\right] \tag{35}$$

and in a steep function case

$$\Gamma(t) = \int_{t_s}^{t}\Omega(t)\,\mathrm{d}t + \arctan\left\{\frac{\Omega_1}{\Omega(t)}\tan[(t_s-t_0)\Omega_1]\right\}. \tag{36}$$

This result is to be expected because the solution of the TDHO equation (1) in a constant parameter region is sinusoidal. This assertion is made clear by $\mathbb{R}e\{x\} = g(t)\cos\gamma(t) = G(t)\cos\Gamma(t)$. In a constant parameter region, the amplitude $G(t)$ is time independent and the phase $\Gamma(t)$ has a linear dependence on time. In Fig. 5, the time dependent amplitude $g(t)$ and the nonlinear phase $\cos\gamma(t)$ are plotted together with their product which yield a sinusoidal function of the form $G\cos\Gamma$.

The amplitude function $G(t)$ does not satisfy the Ermakov equation in the entire time domain. Nonetheless, this expression satisfies an Ermakov type equation in each interval provided that the invariant is appropriately modified. To wit, in interval I, $G^2(t_1) = a_1^2$ and the Ermakov invariant is thus $Q_{\mathrm{I}} = a_1^2\Omega_1$. Whereas in interval II, $G^2(t_2) = a_1^2[(\Omega_1^2/\Omega_2^2)\sin^2(t_s\Omega_1) + \cos^2(t_s\Omega_1)]$ which leads to a different invariant $Q_{\mathrm{II}} = a_1^2\Omega_2[(\Omega_1^2/\Omega_2^2)\sin^2(t_s\Omega_1) + \cos^2(t_s\Omega_1)]$. In contrast, *the amplitude $g(t)$ does satisfy the Ermakov equation in the whole time domain. The invariant Q, then remains a constant of motion throughout the intervals*. From this constant, it is possible to establish a general relationship between the phase independent amplitudes as shown below.

6.1. Amplitudes ratio

At constant parameter regions, the sinusoidal motion of an oscillator is usually made up from a constant amplitude and a linear phase. Therefore, measurements often correspond to G and Γ. Nonetheless, we have attempted to make

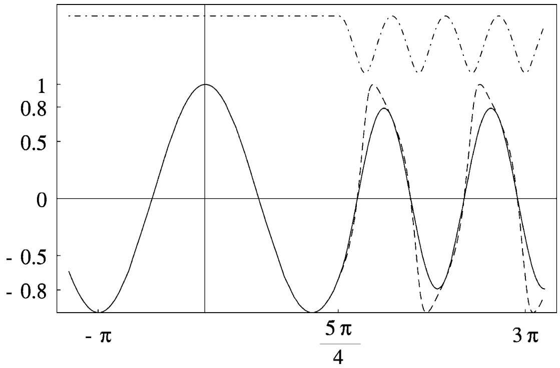

Fig. 5. Time dependent amplitude function $g(t)$ (dash dotted line), cosine of the nonlinear phase $\gamma(t)$ (dotted line), and their product (solid line).

clear in the previous section that such a choice is not unique. The sinusoidal motion may equally be composed from a time dependent amplitude $g(t)$ and a nonlinear phase $\gamma(t)$. Let us evaluate the relationship between amplitudes in the usual $G\cos\Gamma$ representation. The phase independent amplitudes in the intervals I and II are

$$G_1 = a_1, \qquad G_2 = a_2\sqrt{1+\sigma^2+2\sigma\cos(2\theta)}, \tag{37}$$

where

$$\sigma = \frac{b_2}{a_2}. \tag{38}$$

Substitution of the amplitudes a_1 and a_2 from (37) in the invariant relationship (13) yields

$$G_1^2\Omega_1 = \frac{G_2^2(1-\sigma^2)}{1+\sigma^2+2\sigma\cos(2\theta)}\Omega_2. \tag{39}$$

This equation states the relation that should be observed by the phase independent amplitudes. It is valid at constant parameter intervals regardless of the way in which the time dependent parameter varies from one steady state to another. This is so because the sub-period condition is not necessary to establish this result since the amplitude dependence on Ω, i.e. $\sigma(\Omega)$ has not been invoked in the derivation. Let us now consider three different cases, depicted in Fig. 6, where a step function takes place at different moments of the oscillation.

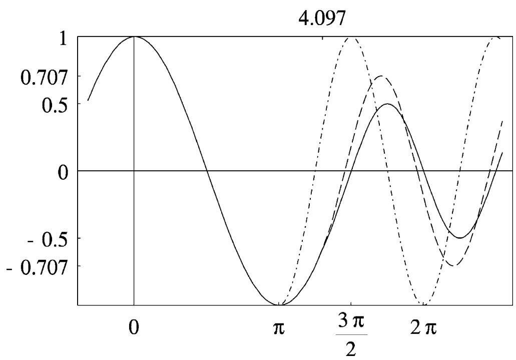

Fig. 6. Step at $t_s = \pi + 0$ (dash dotted line), $t_s = \pi + (1/2)\arccos\sigma \simeq 4.0969$ (dashed line), and $t_s = \pi + (\pi/2)$ (solid line).

- If a step takes place when the position is at a maximum or a minimum, then $\theta = t_s - t_0 = m\pi$ where m is an integer. The invariant relationship for the phase independent amplitudes (39) then reads

$$\frac{G_2^2}{G_1^2} = \frac{\Omega_1}{\Omega_2}\frac{(1+\sigma)^2}{(1-\sigma^2)} = \frac{\Omega_1}{\Omega_2}\frac{(1+\sigma)}{(1-\sigma)}. \tag{40}$$

  In interval II $\sigma = (\Omega_2 - \Omega_1)/(\Omega_2 + \Omega_1)$ and the amplitude ratio is then

$$\frac{G_2^2}{G_1^2} = \frac{\Omega_1}{\Omega_2}\frac{\Omega_2}{\Omega_1} = 1, \tag{41}$$

  that is, the amplitude remains unchanged if the step takes place at maximum displacement.
- On the other hand, if the step takes place at zero displacement $\theta = t_s - t_0 = m(\pi/2)$, i.e. at the equilibrium position, the invariant relationship is then

$$\frac{G_2^2}{G_1^2} = \frac{\Omega_1}{\Omega_2}\frac{(1-\sigma)^2}{(1-\sigma^2)} = \frac{\Omega_1}{\Omega_2}\frac{(1-\sigma)}{(1+\sigma)} = \left(\frac{\Omega_1}{\Omega_2}\right)^2. \tag{42}$$

  The amplitudes ratio is then proportional to the inverse ratio of the frequencies:

$$\frac{G_2}{G_1} = \frac{\Omega_1}{\Omega_2}. \tag{43}$$

  This case preserves the energy of the oscillator since for a unit mass $(1/2)G_1^2\Omega_1^2 = (1/2)G_2^2\Omega_2^2$.
- The adiabatic condition requires the time dependent parameter to vary slowly compared with the fundamental frequency of the system. The adiabatic invariant derived under these circumstances is $I = \mathcal{E}/\Omega(t)$ where $\mathcal{E}$ is the energy of the oscillator. The orthogonal functions invariant has been shown to become the adiabatic invariant in the slow variation limit [14]. The adiabatic relationship is obtained from (39) when $\sigma = 0$. Nonetheless, it is possible to obtain the same amplitude ratio in the abrupt case if the quotient $(1-\sigma^2)/(1+\sigma^2+2\sigma\cos(2\theta))$ is equal to one. This condition is fulfilled if

$$-\sigma = \cos(2\theta) \Rightarrow \theta = \tfrac{1}{2}\arccos\sigma. \tag{44}$$

The amplitude then decreases due to the step function to the same value that would be obtained in the adiabatic case, namely $\sqrt{\Omega_1/\Omega_2}$. If the step is very small $(\Omega_2 - \Omega_1) \ll 1$, then $\arccos[(\Omega_2-\Omega_1)/(\Omega_2+\Omega_1)] = \pi/2$ and $\theta = \pi/4$.

The amplitude ratio lies between the limits

$$1 \le \frac{G_2}{G_1} \le \frac{\Omega_1}{\Omega_2}, \tag{45}$$

that is, for a step with height β, the amplitude G_2 must be between 1 and $1/(1+\beta)$ times the initial amplitude. If β is negative, the step is downwards. The same results hold in this case although it should be clear that if the step decreases to zero the amplitude diverges and the motion is no longer bounded.

7. Conclusions

Approximate analytical solutions to the TDHO equation have been presented for a sub-period time varying parameter with an initial constant amplitude. The new solutions are not in the class of the usual WKB results since they do not require slowly varying parameters. The solution has been expressed in terms of amplitude and phase variables in two different ways. The representation $x(t) = g(t)\cos\gamma(t)$ allows the amplitude (20) and phase (28) functions to satisfy, within the approximation, the Ermakov and the nonlinear frequency equations as well as the TDHO equation. These results seem particularly valuable for problems involving an auxiliary equation of this form [15]. On the other hand, the representation $x(t) = G(t)\cos\Gamma(t)$ has a constant amplitude and a linear phase for time independent parameter intervals. The general relationship between the amplitude and phase in this latter representation has been established through the orthogonal functions invariant. The solution has been shown to be adequate for a time parameter that varies monotonically in times much shorter than the fundamental period of the system but not necessarily infinitesimal. The explicit time dependence of the parameter may be chosen from a variety of functions, a hyperbolic tangent dependence has been used in the examples presented here. It is worth mentioning that the singularities often introduced by piecewise integration do not occur in the solutions presented here.

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