ORTHOGONAL FUNCTIONS EXACT INVARIANT AND THE ADIABATIC LIMIT FOR TIME DEPENDENT HARMONIC OSCILLATORS

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Abstract

An invariant arising from the linearly independent solutions is evaluated for the time dependent harmonic oscillator equation. The relationship of this quantity with the Lewis invariant is established. The physical meaning of the procedure is stressed. In the adiabatic approximation, the proposed invariant turns into the well known relationship where the energy is proportional to the frequency.

Keywords: Invariants, Time dependent oscillator.

Introduction

The energy of an oscillator with time dependent parameters is well known to be no longer a constant of motion. Nonetheless, it is still possible to find invariants of motion for this system. The first approach to this problem was developed through the theory of adiabatic invariants [1]. This procedure requires the time evolution of the parameters to be slow compared with the characteristic period of the time independent parameters. The adiabatic invariant for a system executing periodic motion with slowly varying parameters is given by the area integral in

phase space:

$$I_{adiab} = \frac{1}{2\pi} \oint dpdq, \tag{1}$$

where q is a canonical position coordinate and p its corresponding momentum. In particular, the adiabatic invariant for a one dimensional time dependent oscillator is equal to the ratio of its energy over frequency $I_{adiab} = \frac{E}{\omega}$. This result was first derived by Ehrenfest [2] in relation with Planck's oscillators. Einstein at Solvay, also referred to this constant ratio for a pendulum with slowly varying length [3]. The adiabatic approach has been used extensively in the averaging of Hamiltonian systems.

In recent decades, an *exact* invariant for the time dependent harmonic oscillator has been obtained by invoking Kruskal's theory [4], dynamical algebra, transformation-group theory, Noether's theorem and Ermakov's method. These methods have been shown to yield equivalent results [5]. The Lewis invariant thus derived is:

$$I = \frac{1}{2}[\frac{q^2}{\rho^2} + (\rho\dot{q} - q\dot{\rho})^2], \tag{2}$$

where ρ is an auxiliary function of time that satisfies the Ermakov equation $\ddot{\rho} + \Omega^2(t)\rho = \frac{1}{\rho^3}$. A physical meaning of this invariant has been proposed by introducing an auxiliary plane of motion [6]. The actual one dimensional motion is then viewed as a projection of a two dimensional motion. The invariant is then proportional to the angular momentum squared in this ancillary two dimensional plane.

In the present communication, we derive a different conserved quantity, albeit closely related with the Lewis invariant. The procedure is extremely simple and involves only elementary algebra. The proposed invariant is then expressed using different variables in order to clarify the physical meaning of the involved quantities. Finally, the adiabatic limit of this exact invariant is shown to reproduce well known results of adiabatic theory.

1. Orthogonal functions invariant derivation

Let us commence with the time dependent harmonic oscillator differential equation:

$$\ddot{q} + \Omega^2(t)q = 0, \tag{3}$$

where q is a canonical coordinate and $\Omega(t)$ is an arbitrary continuous function of time. Let two solutions to the differential equation be q_1 and q_2. The product of q_1 times the differential equation of q_2 yields

$$q_1\ddot{q}_2 + \Omega^2 q_1 q_2 = 0,$$

whereas the product of q_2 times the differential equation of q_1 yields

$$q_2\ddot{q}_1 + \Omega^2 q_1 q_2 = 0.$$

The difference between these two equations gives:

$$q_1\ddot{q}_2 - q_2\ddot{q}_1 = 0$$

This equation may be written as a first order derivative $\frac{d}{dt}(q_1\dot{q}_2 - q_2\dot{q}_1)$ and thus there exists an exact invariant given by:

$$Q = q_1\dot{q}_2 - q_2\dot{q}_1 \tag{4}$$

This invariant corresponds to the Wronskian and may be used to obtain a second linearly independent solution provided that one particular solution is known [7]. From this expression it is clear that if the function q_2 is proportional to q_1 the invariant is zero. Since both functions satisfy the same differential equation, non vanishing contributions must come from the linearly independent solutions. In order to show this assertion explicitly, we rewrite the orthogonal functions invariant as:

$$Q \equiv q\dot{q}_\perp - q_\perp\dot{q} \tag{5}$$

where the independent solutions are defined by the usual Sturm Liouville orthogonal functions integral $\int qq_\perp dt = 0$.

1.1. Representation in terms of amplitude and phase variables

Allow for the independent solutions of the differential equation to be written in terms of time dependent amplitude (A) and phase (s) variables:

$$q = cA\cos s, \quad q_\perp = \pm c_\perp A\sin s, \tag{6}$$

where c and $c_\perp$ are amplitude constants of the orthogonal functions. It is a matter of convention whether the function $q_\perp$ is defined with the plus or minus sign. The orthogonal functions invariant in terms of these variables becomes:

$$Q = \pm cc_\perp A^2\frac{ds}{dt} \tag{7}$$

This quantity therefore establishes a relationship between the squared amplitude and the frequency of the motion. It may be positive or negative depending on the sign convention for $q_\perp$ and the sign of the derivative of the phase. Q is negative if the function q leads $q_\perp$ as shown in figure 1. This condition is achieved, for example, if $q_\perp$ is defined with the

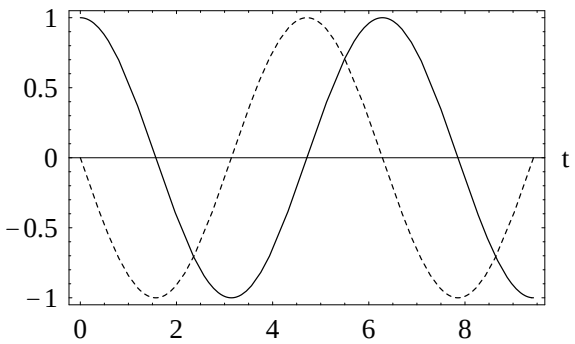

Figure 1. Function q (solid line) leads $q_\perp$(dotted line); constant amplitude functions with linear time dependent phase are illustrated; $Q < 0$.

negative sign and $\frac{ds}{dt} > 0$. On the other hand, any of the two solutions may be taken as the reference function q and the other as the orthogonal function $q_\perp$. If the second solution in eq. (4) is taken as the reference function, the invariant reads $Q_{21} = q_\perp \dot{q} - q\dot{q}_\perp = -Q$. The change of sign shows that the leading function becomes the lagging function if the reference and orthogonal functions are interchanged.

Time reversal changes the sign of the invariant reflecting the fact that the leading field becomes the lagging field upon time inversion provided that the amplitude constants remain unaltered under this operation.

1.2. Relationship with Lewis invariant

Allow for the invariant to be rewritten in terms of the amplitude and the position coordinate q. To this end, the orthogonal solution is written as $q_\perp = c_\perp A\sqrt{1-(\frac{q}{cA})^2}$. Substitution in the invariant definition equation (5) yields $Q = c_\perp(q\dot{A} - A\dot{q})\left(1-(\frac{q}{cA})^2\right)^{-\frac{1}{2}}$. The square of this expression is rearranged to obtain $\frac{Q^2}{c_\perp^2} = \left(\frac{Q}{cc_\perp}\right)^2 \frac{q^2}{A^2} + (q\dot{A} - A\dot{q})^2$. A constant $h = \frac{Q}{cc_\perp}$ is then introduced and allowed to satisfy $h^2 =$

$1-(c_\perp^2/c^2)$. Substitution in the previous equation leads to:

$$\frac{c^2-c_\perp^2}{2}=\frac{1}{2}\left(h^2\frac{q^2}{A^2}+(q\dot{A}-A\dot{q})^2\right), \tag{8}$$

but this is precisely the Lewis invariant in eq. (2), which has been previously derived using rather more complex mathematical methods. The constant $h\neq 1$ is a subsequent generalization proposed by Eliezer and Gray [6]. This invariant $I=\frac{1}{2}(c^2-c_\perp^2)$ therefore represents a measure of the difference between the squared amplitude constants of the orthogonal functions.

1.3. Differential equation general solution

Consider, the general case where the solution is constructed from the linear superposition of the orthogonal solutions:

$$q_g=c\,(c_1A\cos s+c_2A\sin s) \tag{9}$$

where c_1 and c_2 are the linear combination constants. The orthogonal function of this general solution is $q_{g\perp}=q_g\int\frac{Q}{q_g^2}dt$. Substitution of the invariant (7) within the integral permits the straight forward evaluation over ds.

$$q_{g\perp}=\mp c_\perp\,(c_2A\cos s-c_1A\sin s) \tag{10}$$

where it is clear that the amplitude constants between orthogonal functions $c, c_\perp$ play an altogether different role from the linear combination constants c_1, c_2.

This result is perhaps better visualized in terms of the phase shift between the two functions. Let the general case be written as:

$$q_g=cc_0A\cos(s+\varphi_0) \tag{11}$$

where c_0 is a constant amplitude factor and φ_0 is an arbitrary constant phase; the relationship with the linear combination constants is clearly $c_1=c_0\cos\varphi_0$ and $c_2=-c_0\sin\varphi_0$. The orthogonal solution is given by the 90° out of phase function:

$$q_{g\perp}=c_\perp c_0A\cos(s+\varphi_0\mp\frac{\pi}{2})=\pm c_\perp c_0A\sin(s+\varphi_0). \tag{12}$$

The orthogonal function is of course the same using either the integration or the phase shift procedure. The invariant in this general case is then $Q=\pm cc_\perp c_0^2A^2\frac{ds}{dt}$.

It is of utmost relevance to recognize that two orthogonal functions, namely q and $q_\perp$ are being requested to simultaneously satisfy the time

dependent harmonic oscillator differential equation. These orthogonal solutions are being used in a rather different fashion than the usual additive way invoked in order to construct the general solution.

2. complex algebra representation

The previous results may also be tackled using a complex formalism [8]. To this end, consider the complex polar representation of the canonical coordinate $\tilde{q} = Ae^{is}$ where A and s are real variables. The time dependent harmonic oscillator equation (3) unfolds onto:

$$\frac{d^2A}{dt^2} - A\left(\frac{ds}{dt}\right)^2 = -\Omega^2 A, \tag{13}$$

$$2i\frac{dA}{dt}\frac{ds}{dt} + iA\frac{d^2s}{dt^2} = 0, \tag{14}$$

where Ω^2 is restricted to be a real function. From the last equation, provided that A is not zero, we obtain the previously derived invariant $Q = A^2\frac{ds}{dt}$ with $c = c_\perp = 1$. With the aid of this result, it is possible to write separate equations for the amplitude and the phase:

$$\frac{d^2A}{dt^2} - \frac{Q^2}{A^3} = -\Omega^2 A, \tag{15}$$

$$\left(\frac{ds}{dt}\right)^4 - \frac{3}{4}\left(\frac{d^2s}{dt^2}\right)^2 + \frac{1}{2}\frac{ds}{dt}\frac{d^3s}{dt^3} = \Omega^2\left(\frac{ds}{dt}\right)^2, \tag{16}$$

the former of these equations is the Ermakov equation. The auxiliary function ρ, mentioned in the introduction, then represents the amplitude of the motion when the trajectory is described in terms of amplitude and phase variables.

The general solution in the complex formalism is given by the linear superposition of two terms with opposite phases:

$$\tilde{q_g} = A_+e^{is_+} + \sigma A_+e^{is_+} \tag{17}$$

This result may be rewritten in the additive representation as:

$$\tilde{q_g} = (1+\sigma)A_+\cos s_+ + i(1-\sigma)A_+\sin s_+ \tag{18}$$

The invariant, obtained by turning back to the polar representation, is:

$$Q = (1-\sigma^2)A_+^2\frac{ds_+}{dt} \tag{19}$$

In this representation, the invariant is then proportional to the difference between the squared amplitude of the opposite phase functions. In a

one dimensional propagation problem these opposite phases correspond to counter propagating waves [9]. Comparison with the results of the previous section shows that the real part of the complex function $\tilde{q}$ plays the role of q and the imaginary part the orthogonal solution $q_\perp$. The relationship between amplitude constants is then $c = 1 + \sigma$ and $c_\perp = 1 - \sigma$. The Lewis invariant $I = \frac{1}{2}(c^2 - c_\perp^2) = 2\sigma$ may also be obtained using this formalism. According with the linear superposition equation, it represents twice the amplitude ratio of the opposite phase complex solutions. It is worth mentioning that the complex derivation requires a real time dependent parameter Ω^2 whereas the orthogonal functions derivation allows this parameter to be complex.

3. Adiabatic regime

Adiabatic theory makes use of the slowly varying parameters restriction from the outset in order to derive invariants within this approximation. On the other hand, exact invariants derived either with the present method or other schemes are valid even for arbitrarily fast time dependent parameters. An exact invariant should reproduce the corresponding adiabatic invariant in the slowly varying parameters limit. Let the frequency ω be defined by the derivative of the phase so that $s = \int \omega dt$. This expression holds true for an arbitrary frequency time dependence. Care should be taken not to define $s = \omega(t)t$ because then $\dot{s} = \dot{\omega}(t)t + \omega(t)$ which is only roughly equal to the frequency for slow frequency variation and sufficiently short times[1]. The orthogonal functions invariant is then:

$$Q = A^2(t)\omega(t) \tag{20}$$

The energy of a system executing small oscillations with constant frequency is given by $W(\neq t) = \frac{1}{2}mA^2(\neq t)\omega^2(\neq t)$ where m stands for the mass of the object. Consider the time dependent parameters to vary slowly compared with the frequency of oscillation. Under these approximations, the time dependent energy of the oscillator may be estimated with the same formal dependence on the variables but allowing them to be time dependent as well; *i.e.* $W(t) = \frac{1}{2}mA^2(t)\omega^2(t)$. The orthogonal functions invariant is then approximately given by:

$$Q \approx \frac{2}{m}\frac{W(t)}{\omega(t)} \tag{21}$$

This expression, except for the constant $\frac{2}{m}$, is equal to the invariant derived from adiabatic theory. It corresponds to the action variable

[1]Remark made to the author (MFG) by professor N. G. van Kampen during the talk.

in a canonical transformation where the invariant is taken as the new momentum.

4. Final remarks

The orthogonal functions invariant is an exact constant of motion enven for arbitrarily fast time dependent parameters. Its representation in amplitude and phase variables establishes the invariance of the square amplitude times the frequency. Straight forward manipulation leads to the Lewis invariant. Besides the simplicity of the derivation, the procedure elucidates the physical meaning of the involved quantities. To wit, the positive or negative sign of the invariant exhibits whether the orthogonal function is lagging or leading the reference function. The auxiliary equation variable represents the amplitude of the motion. The Lewis invariant may be viewed as the difference between the squared amplitude constants of the orthogonal functions without recurring to an auxiliary plane. Finally, the orthogonal functions invariant is the exact result corresponding to the adiabatic invariant ratio of energy over frequency.

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