# Amplitude and phase representation of monochromatic fields in physical optics

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The conservation equation for a monochromatic field with arbitrary polarization propagating in an inhomogeneous transparent medium is expressed in terms of amplitude and phase variables. The expressions obtained for linearly polarized fields are compared with the results obtained in the eikonal approximation. The electric field wave equation is written in terms of intensity and phase variables. The transport equations for the irradiance and the phase are shown to be particular cases of these derivations. The conservation equation arising from the second-order differential wave equation is shown to be equivalent to that obtained from Poynting's theorem. 

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## 1. INTRODUCTION

Different types of interferometric schemes have been the major technique used to obtain phase information from a wide variety of wave phenomena. Diffraction-limited resolution in optical instruments is attainable with this approach.[1] However, the experimental setup usually requires stable, well-aligned systems together with an adequate coherence of the source. Nonetheless, certain arrangements overcome some of these stringent experimental constraints: for example, common path schemes[2] and interferometers involving nonlinear elements such as phase-conjugate mirrors[3] or self-aligning filters.[4]

Noninterferometric optical testing is usually performed in the framework of geometrical optics through the tracing of pencils of rays. The wave-front or isophase curves are then reconstructed from the transverse aberration data.[5] The main advantages of this method are that the experimental conditions are usually robust and economical, the light source may be coherent or incoherent, and no reference surfaces are required.[6] However, the resolution obtained with these techniques is limited by the eikonal approximation.[7,8] Furthermore, image reconstruction of phase objects is in most cases impractical using these procedures.

An alternative proposal developed in the last two decades has been to measure the irradiance in two planes orthogonal to the propagation direction in order to retrieve the phase.[9] The undemanding experimental conditions required to obtain the intensity data enhance the range of applicability of this technique. For this reason, the retrieval of the phase from intensity measurements has received much interest in different fields of science ranging from wave-front sensing in optical testing[10] and adaptive optics[11] to visualization of phase objects in different regions of the electromagnetic spectrum.[12]

The deterministic approach to phase retrieval from intensity records is based on the irradiance transport equation. This result was first derived by using a parabolic equation as a starting point.[13] Poynting's theorem, together with the representation of Poynting's vector in the eikonal approximation, has also been used in order to obtain this transport equation.[14]

In this paper, we show that the amplitude and phase representation of Poynting's vector may be obtained from Maxwell's equations under rather general considerations. The conservation equation in terms of these variables is presented for inhomogeneous transparent linear media without invoking the paraxial approximation or the short-wavelength limit. These results are compared with the equations derived in the geometrical-optics limit and the well-known case of plane-wave propagation in free space. The paraxial limit of the general expressions is then considered in order to obtain the intensity transport equation for inhomogeneous media. The homogeneous restriction is shown to reproduce previous results.

The amplitude and phase representation is then assessed by using the electric field wave equation as a starting point. This representation leads to two coupled second-order differential equations that may be decoupled only under restricted circumstances. These equations are written in terms of the intensity (rather than amplitude) and phase variables, since the intensity is the quan-

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tity that is actually measured in most circumstances. The equation arising from the imaginary part of the wave equation is shown to be equivalent to the conservation equation derived from Poynting's theorem in the case of a monochromatic field.

## 2. POYNTING'S CONSERVATION EQUATION

Consider the propagation of an electromagnetic wave in an inhomogeneous isotropic linear medium. Poynting's theorem for complex fields in dispersionless linear media without absorption is given by[15]

$$\nabla \cdot \tfrac{1}{2}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) + \frac{\partial}{\partial t}[\tfrac{1}{2}(\mathbf{B} \cdot \mathbf{H}^* + \mathbf{E} \cdot \mathbf{D}^*)] = -\tfrac{1}{2}(\mathbf{J}^* \cdot \mathbf{E} + \mathbf{J} \cdot \mathbf{E}^*), \quad (1)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{D}$ and $\mathbf{H}$ are their corresponding displacements, and $\mathbf{J}$ is the current. This continuity equation ensures that the Poynting vector for complex fields,

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}), \quad (2)$$

is a real quantity, in contrast with the complex definition often used in the literature.[16] The electric field may be written in the polar representation with a complex vector amplitude $\tilde{\mathbf{a}}(\mathbf{r}, t)$ and a real scalar phase $\phi(\mathbf{r}, t)$ as $\mathbf{E}_g(\mathbf{r}, t) = \tilde{\mathbf{a}}(\mathbf{r}, t)\exp[i\phi(\mathbf{r}, t)]$. In the Fourier domain, the field is expressed in terms of infinite wave trains with fixed frequency $\omega$, where each component satisfies Maxwell's equations because of the linearity imposed on the constitutive relationships. Each component is then a monochromatic wave:

$$\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{a}}(\mathbf{r})\exp\{i[\phi(\mathbf{r}) - \omega t]\}. \quad (3)$$

The magnetic field corresponding to each electric field component may be evaluated from Maxwell's induction equation $\mathbf{B} = -\int \nabla \times \mathbf{E} dt$. The magnetic displacement for a monochromatic wave train is then given by

$$\mathbf{H} = -\frac{1}{\omega\mu}(\tilde{\mathbf{a}} \times \nabla\phi + i\nabla \times \tilde{\mathbf{a}})\exp\{i[\phi(\mathbf{r}) - \omega t]\}, \quad (4)$$

where $\mu$ is the medium permeability. The Poynting vector (2) in the amplitude and phase representation is then

$$\mathbf{S} = -\frac{1}{2\omega\mu}[\tilde{\mathbf{a}} \times (\tilde{\mathbf{a}}^* \times \nabla\phi) - i\tilde{\mathbf{a}} \times (\nabla \times \tilde{\mathbf{a}}^*) + \text{c.c.}], \quad (5)$$

where c.c. stands for complex conjugate. The triple vector product may be rewritten in terms of outer products to yield

$$\mathbf{S} = \frac{1}{\omega\mu}(\tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}}^*)\nabla\phi + \frac{1}{2\omega\mu}[-(\tilde{\mathbf{a}} \cdot \nabla\phi)\tilde{\mathbf{a}}^* + i\tilde{\mathbf{a}} \times (\nabla \times \tilde{\mathbf{a}}^*) + \text{c.c.}]. \quad (6)$$

The normalized intensity Poynting vector $\mathbf{S}_N = \mathbf{S}/(\tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}}^*)$ has the form of the gradient of a scalar function $\phi$ plus a vector field that resembles the Helmholtz decomposition proposed by Paganin and Nugent.[17] To establish an exact equivalence, it would be necessary to identify the vector field part that is divergence free. It is worth pointing out that this derivation does not impose the scalar theory condition of a scalar wave field with its concomitant definition of flow.[18,19] The energy density

$$u = \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{H}^* + \mathbf{E} \cdot \mathbf{D}^*) \quad (7)$$

is time independent for a time-harmonic field, and its time derivative vanishes in the conservation equation. The Poynting theorem for a monochromatic vector wave of arbitrary form propagating in linear inhomogeneous media without charges or absorption reads as

$$\nabla \cdot \left\{\frac{1}{\omega\mu}(\tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}}^*)\nabla\phi + \frac{1}{2\omega\mu}[-(\tilde{\mathbf{a}} \cdot \nabla\phi)\tilde{\mathbf{a}}^* + i\tilde{\mathbf{a}} \times (\nabla \times \tilde{\mathbf{a}}^*) + \text{c.c.}] = 0. \quad (8)$$

From this expression, it follows that in order to derive the phase, up to a constant, it is mathematically necessary to know the complex vector amplitude $\tilde{\mathbf{a}}(\mathbf{r})$ spatial dependence. The complex nature of the amplitude in this context refers to the state of polarization of the wave.[20] Let us impose the restriction of a real amplitude $\mathbf{a}(\mathbf{r})$, which implies that only linearly polarized fields are allowed. The Poynting vector is then $\mathbf{S} = (1/\omega\mu)\times[(\mathbf{a} \cdot \mathbf{a})\nabla\phi - (\mathbf{a} \cdot \nabla\phi)\mathbf{a}]$. The term $\tilde{\mathbf{a}} \times (\nabla \times \tilde{\mathbf{a}}^*)$ is therefore relevant only in the presence of elliptically polarized light and is possibly connected with vortex structures. From the first of Maxwell's equations, $\nabla \cdot \mathbf{E} = -\mathbf{E} \cdot \nabla \ln \epsilon + \rho/\epsilon$, and in the absence of sources,

$$\nabla \cdot \mathbf{a} + i\mathbf{a} \cdot \nabla\phi = -\mathbf{a} \cdot \nabla \ln \epsilon, \quad (9)$$

but since $\epsilon$ is real for a nonabsorbing medium, the term $\mathbf{a} \cdot \nabla\phi = 0$ and the flow is thus

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{a} \cdot \mathbf{a})\frac{\nabla\phi}{\omega}. \quad (10)$$

This expression is the general form of Poynting's vector in terms of amplitude and phase variables for a linearly polarized wave propagating in an inhomogeneous transparent medium. Poynting's theorem for a linearly polarized monochromatic wave with arbitrary wave front reads as

$$\nabla \cdot \left(\frac{1}{\mu\omega}\mathbf{a} \cdot \mathbf{a}\nabla\phi\right) = 0. \quad (11)$$

If we rewrite the phase in terms of the optical path, i.e., $\phi(\mathbf{r}) = k_0 W(\mathbf{r})$, Poynting's vector is

$$\mathbf{S} = \frac{I}{\mu c}\nabla W, \quad (12)$$

where the intensity is defined as $I = \mathbf{a} \cdot \mathbf{a}$ and $\omega/k_0 = c$. The conservation equation may be expanded to yield

$$I\nabla^2 W + \nabla W \cdot \nabla I - I\nabla W \cdot \nabla \ln \mu = 0. \quad (13)$$

Both Poynting's vector (12) and the conservation equation (13) have been derived without making any assumptions on the wave-vector magnitude. However, these expressions have long been known in the short-wavelength limit, where the optical path function $W$ satisfies the eikonal equation $\nabla W \cdot \nabla W = \epsilon\mu$. The transport equation (13) is also obtained in geometrical optics from the second-order wave equations in the limit where only linear terms in the wavelength are considered (only **L** terms are retained in Born and Wolf's terminology). However, the eikonal derivation neglects the amplitude derivatives and the inhomogeneity of the medium on a wavelength scale. As is well-known, such an approximation describes refractive phenomena but does not model wave phenomena such as interference and diffraction or propagation through an abrupt interface.

In the present derivation, these equations have been obtained without invoking the short-wavelength limit. The price that we have paid is that the amplitudes must be real. Therefore Poynting's vector and the conservation equation in the forms (12) and (13), respectively, are exact for linearly polarized wave fields but correspond to the short-wavelength approximation for fields with arbitrary elliptical polarization. The corresponding equations that are exact for vector waves with arbitrary form are given by Eqs. (6) and (8).

In the particular case of a plane wave with constant electric field amplitude $a_0$ and wave vector $\mathbf{k}$, propagating in a homogeneous medium with relative permittivity $\epsilon_r$, Poynting's vector is given by[16]

$$\mathbf{S} = \sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_r}a_0^2\hat{n}, \tag{14}$$

where $\hat{n}$ is a unit vector in the direction of propagation. This familiar result is a particular case of Eq. (10) for a nonmagnetic medium, where $\nabla\phi/\omega = (k/\omega)\hat{n} = \sqrt{\mu_0\epsilon_0\epsilon_r}\hat{n}$.

## 3. IRRADIANCE PROPAGATION EQUATION

The time-independent conservation equation may be rewritten in the form of a propagation equation. To this end, let the gradient operator be separated into a transverse and a longitudinal operator: $\nabla = \nabla_T + (\partial/\partial z)\hat{e}_z$. The transverse part in Cartesian or cylindrical coordinates is correspondingly given by $\nabla_T = (\partial/\partial x)\hat{e}_x + (\partial/\partial y)\hat{e}_y$ or $\nabla_T = (\partial/\partial\rho)\hat{e}_p + (1/\rho)(\partial/\partial\theta)\hat{e}_\theta$. The divergence equation (11) for a nonmagnetic medium then resembles the form of a continuity equation:

$$\nabla_T \cdot (I\nabla_T\phi) + \frac{\partial}{\partial z}\left(I\frac{\partial\phi}{\partial z}\right) = 0, \tag{15}$$

where $z$ plays the role of $t$, as is customary in the Hamiltonian analogy between mechanics and optics.[20] The wave-vector magnitude squared is defined by

$$\nabla\phi \cdot \nabla\phi = k^2 = \nabla_T\phi \cdot \nabla_T\phi + (\partial\phi/\partial z)^2. \tag{16}$$

This definition in terms of the wave's phase subordinates the relationship $\mathbf{k} \cdot \mathbf{k} = \mu\epsilon\omega^2$ as an approximate expression that is valid in the eikonal limit. The derivative of the phase in the $z$ direction is then $\partial\phi/\partial z = (k^2 - \nabla_T\phi \cdot \nabla_T\phi)^{1/2}$. Let the wave propagation be preferentially in the $z$ direction in order to impose the paraxial approximation

$$\left(\frac{\partial\phi}{\partial z}\right)^2 \gg \nabla_T\phi \cdot \nabla_T\phi, \tag{17}$$

so that the binomial expansion of the root yields

$$\frac{\partial\phi}{\partial z} = k - \frac{1}{2k}\nabla_T\phi \cdot \nabla_T\phi + \cdots. \tag{18}$$

Retaining only the first term of the above expression in the continuity equation (15) yields

$$\nabla_T \cdot (I\nabla_T\phi) + k\frac{\partial I}{\partial z} + I\frac{\partial k}{\partial z} = 0. \tag{19}$$

This equation represents the propagation of the irradiance in an inhomogeneous medium in the paraxial approximation. *In vacuo*, the wave-vector magnitude is constant, provided that diffraction or interference effects may be neglected. Within the present approximation, we have $\partial k/\partial z = 0$, which amounts to neglecting second-order derivatives of the phase in the direction of propagation. The irradiance transport equation then reads as

$$I\nabla_T^2\phi + \nabla_T I \cdot \nabla_T\phi + k\frac{\partial I}{\partial z} = 0. \tag{20}$$

This particular result corresponds to that previously obtained by Teague[21] and subsequent workers.

## 4. WAVE EQUATION IN AMPLITUDE AND PHASE VARIABLES

To obtain the field propagation in an inhomogeneous medium, it is necessary to solve the corresponding wave equation. To this end, consider the electric field wave equation that arises from Maxwell's equations for a linear isotropic dispersionless medium without charges in SI units:

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = -\nabla(\mathbf{E} \cdot \nabla \ln \epsilon) - \nabla \ln \mu(\nabla \times \mathbf{E}). \tag{21}$$

Allow for an inhomogeneous nonabsorbing material so that the permittivity and the permeability are real space-dependent quantities. Let the electric field be written in terms of the complex function (3) but with the restriction of a linear polarization, which implies a real amplitude $\mathbf{a}(\mathbf{r})$. The real part of the wave equation yields

$$\nabla^2\mathbf{a} - (\nabla\phi \cdot \nabla\phi)\mathbf{a} + \mu\epsilon\omega^2\mathbf{a} = F_E, \tag{22}$$

where the electric field source terms due to the inhomogeneity of the medium are

$$\begin{aligned} F_E &= -\nabla(\mathbf{a} \cdot \nabla \ln \epsilon) - \nabla \ln \mu \times \nabla \times \mathbf{a} \\ &= -\nabla[\mathbf{a} \cdot (\nabla \ln \mu\epsilon)] + (\mathbf{a} \cdot \nabla)\nabla \ln \mu + (\nabla \ln \mu \cdot \nabla)\mathbf{a}. \end{aligned} \tag{23}$$

To obtain an equation in terms of the intensity instead of the amplitude, evaluate the outer product of the amplitude $\mathbf{a}$ with Eq. (22):

$$\mathbf{a} \cdot \nabla^2 \mathbf{a} - \mathbf{a} \cdot \mathbf{a}(\nabla \phi \cdot \nabla \phi) + \mu \epsilon \omega^2 \mathbf{a} \cdot \mathbf{a} = F_E \cdot \mathbf{a}. \tag{24}$$

Recall the identity

$$a_m \nabla^2 a_m = \frac{1}{2} \nabla^2 a_m^2 - \frac{1}{4} \frac{1}{a_m^2} \nabla a_m^2 \cdot \nabla a_m^2, \tag{25}$$

where the scalars $a_m$ represent the amplitude components in Cartesian coordinates of the electric field $\mathbf{E} = \Sigma_{m=1}^{3} a_m \exp\{i[\phi(\mathbf{r}) - \omega t]\}\hat{e}_m$. The equation in terms of intensity and phase is then

$$\frac{1}{2}\nabla^2 I - \frac{1}{4}\sum_{m=1}^{3}\frac{1}{I_m}\nabla I_m \cdot \nabla I_m - I(\nabla\phi \cdot \nabla\phi) + \mu_r \epsilon_r k_0^2 I = F_E \cdot \mathbf{a}, \tag{26}$$

where the intensity and its projections are defined by $I = \mathbf{a} \cdot \mathbf{a}$ and $I_m = a_m^2$; the wave-vector magnitude *in vacuo* is $k_0^2 = \mu_0 \epsilon_0 \omega^2$, and the relative permittivity and permeability are defined by $\epsilon = \epsilon_0 \epsilon_r$ and $\mu = \mu_0 \mu_r$. In the case of a scalar field propagating in free space, we obtain

$$\frac{1}{2}\nabla^2 I - \frac{1}{4}\frac{1}{I}\nabla I \cdot \nabla I - I(\nabla\phi \cdot \nabla\phi) + k_0^2 I = 0. \tag{27}$$

If the amplitude (or intensity) of the wave is independent of the wave vector $k_0$, then the above equation must be satisfied by the first and second pair of terms separately. In this case, the latter pair leads to the eikonal equation. Therefore, in a homogeneous medium, any linearly polarized time-harmonic wave whose amplitude is wave-vector independent obeys rigorously the laws of geometrical optics. The ample validity of the eikonal approximation to waves where the vector potential is associated with a scalar complex potential has been discussed in detail by Green and Wolf.[22]

To obtain a phase transport equation, let the gradient of the phase be written in terms of the transverse and longitudinal operators, i.e., $\nabla\phi \cdot \nabla\phi = \nabla_T\phi \cdot \nabla_T\phi + (\partial\phi/\partial z)^2$; the equation reads as

$$I\left(\frac{\partial\phi}{\partial z}\right)^2 = \frac{1}{2}\nabla^2 I - \frac{1}{4}\frac{1}{I}\nabla I \cdot \nabla I - I\nabla_T\phi \cdot \nabla_T\phi + k_0^2 I. \tag{28}$$

The parabolic approximation derived by Teague is then obtained by approximating the root to first order and neglecting the longitudinal derivatives of the intensity:

$$\left(\frac{\partial\phi}{\partial z}\right) \simeq k_0\left[1 + \frac{1}{2k_0^2 I^2}\left(\frac{1}{2}I\nabla_T^2 I - \frac{1}{4}\nabla_T I \cdot \nabla_T I - I^2\nabla_T\phi \cdot \nabla_T\phi\right)\right]. \tag{29}$$

There is an erratum in Teague's result,[13] as pointed out recently[23]; there is also a minor error in a later paper by Teague,[21] in Eq. (5), where the term on the right-hand side should be $2kI^2(\partial\phi/\partial z)$.

On the other hand, the imaginary part of the wave equation (21) in amplitude and phase variables is

$$\mathbf{a}\nabla^2\phi + 2(\nabla\phi \cdot \nabla)\mathbf{a} = -(\mathbf{a} \cdot \nabla \ln \epsilon)\nabla\phi + \nabla \ln \mu \times (\mathbf{a} \times \nabla\phi). \tag{30}$$

Expanding the triple vector product and writing down the equation for the $m$th component gives

$$\frac{1}{a_m}\nabla \cdot (a_m^2 \nabla\phi) = -(\mathbf{a} \cdot \nabla \ln \mu\epsilon)\frac{\partial\phi}{\partial r_m} + (\nabla \ln \mu \cdot \nabla\phi)a_m, \tag{31}$$

where the identity $2a_m\nabla\phi \cdot \nabla a_m + a_m\nabla^2\phi = (1/a_m)\nabla \cdot (a_m^2\nabla\phi)$ has been used. Multiplying this equation by $a_m$ and summing over components yields

$$\nabla \cdot (\mathbf{a} \cdot \mathbf{a}\nabla\phi) = -(\mathbf{a} \cdot \nabla \ln \mu\epsilon)\nabla\phi \cdot \mathbf{a} + (\nabla \ln \mu \cdot \nabla\phi)\mathbf{a} \cdot \mathbf{a}. \tag{32}$$

However, from the condition of a transparent medium (9), the product $\mathbf{a} \cdot \nabla\phi$ is zero and the first term on the right-hand side of this equation vanishes. On the other hand, the divergence of a scalar $1/\mu$ times a vector $\mathbf{a} \cdot \mathbf{a}\nabla\phi$ is

$$\nabla \cdot \left(\frac{1}{\mu}\mathbf{a} \cdot \mathbf{a}\nabla\phi\right) = \frac{1}{\mu}\nabla \cdot (\mathbf{a} \cdot \mathbf{a}\nabla\phi) + \left(\nabla\frac{1}{\mu} \cdot \nabla\phi\right)\mathbf{a} \cdot \mathbf{a}; \tag{33}$$

but since $\nabla(1/\mu) = -(1/\mu)\nabla\, ln\, \mu$, the imaginary part of the wave equation becomes

$$\nabla \cdot \left(\frac{1}{\mu}\mathbf{a} \cdot \mathbf{a}\nabla\phi\right) = 0. \tag{34}$$

This equation is equal to Eq. (11) except for a factor of $1/\omega$. Nonetheless, since $\omega$ is constant for a monochromatic wave, these two results are equivalent. The conservation equation for a coherent field may thus be derived by using either Poynting's theorem or the wave equation as the starting point. However, if the wave amplitude and frequency are time dependent, then these two procedures do not necessarily lead to the same result. If the paraxial wave equation[24] $\nabla_T\mathbf{u} + 2ik(\partial/\partial z)\mathbf{u} = 0$, with $\mathbf{E}(\mathbf{r}) = \mathbf{u}(\mathbf{r})\exp(ikz)$, is used as the initial equation of the derivation rather than the wave equation (21), then the procedure outlined in this section leads to the irradiance transport equation (20). A derivation beginning with the paraxial wave equation has been reported before,[25] although it actually used the same parabolic equation as that described in Teague's references.

To illustrate the decoupling of the two differential equations obtained for the amplitude and the phase, consider the one-dimensional propagation of the field (say in the $z$ direction) in a nonmagnetic medium. The conservation equation leads to an invariant:

$$Q = \mathbf{a} \cdot \mathbf{a} \frac{\partial \phi}{\partial z}. \tag{35}$$

Such an exact invariant also arises in mechanical problems involving the time-dependent harmonic oscillator.[26] For a nonmagnetic medium, the wave equation for a linearly polarized monochromatic field in terms of the amplitude and phase variables is then

$$\frac{\partial^2 \mathbf{a}}{\partial z^2} - \mathbf{a}\left(\frac{\partial \phi}{\partial z}\right)^2 + \mu_0 \omega^2 \epsilon(z)\mathbf{a} = 0. \tag{36}$$

From these two equations, an equation for the phase or the amplitude may be readily obtained. The amplitude equation is

$$\frac{\partial^2 \mathbf{a}}{\partial z^2} - \frac{\mathbf{a}Q}{(\mathbf{a} \cdot \mathbf{a})^2} + \mu_0 \omega^2 \epsilon(z)\mathbf{a} = 0, \tag{37}$$

which for a scalar amplitude becomes the well-known Ermakov equation. Numerical solutions of this equation exhibit the transition from a smooth refractive-index variation where reflection is negligible to the abrupt case treated in Fresnel equations at normal incidence.[27]

## 5. CONCLUSIONS

Poynting's vector and its corresponding conservation equation have been expressed in terms of vector amplitude and scalar phase variables for monochromatic fields propagating in linear inhomogeneous transparent media. The expressions obtained for an arbitrary elliptical polarization involve terms that may not be expressed as the gradient of a scalar phase. Therefore complete phase retrieval from intensity measurements in an inhomogeneous medium requires the use of a linearly polarized or an unpolarized source. From the present derivation, it is clear that the general validity of phase retrieval from intensity propagation techniques ultimately relies on the energy conservation principle. The form of the conservation equation in amplitude and phase variables obtained for linear polarization without making a short-wavelength-limit approximation has been shown to be equal to that obtained in the geometrical-optics limit for an arbitrary polarization. It should nonetheless be stressed that in the former case the amplitude and the phase must also satisfy the expression derived from the wave equation (22), whereas in the latter case the phase is requested to satisfy the eikonal equation. The irradiance and phase transport equations have been obtained from the conservation and wave equations in the paraxial limit. The irradiance transport equation, previously reported in free-space propagation, has been extended here to inhomogeneous transparent media. These results should prove useful to establish the accuracy of the inferred phase, depending on the experimental schemes. The conservation equation for a monochromatic field has been derived from Poynting's theorem as well as from the imaginary part of the wave equation yielding equivalent results.

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